Towards ellipsoidal representations of the Earth's gravitational field

Michael Schmidt[1], Josef Sebera[2,3], Johannes Bouman[1], and Oliver Fabert[4]

[1] Deutsches Geodätisches Forschungsinstitut der Technischen Universität München (DGFI-TUM)
[2] New Technologies for the Information Society, Faculty of Applied Sciences , University of West Bohemia, Pilsen (NTIS)
[3] Astronomical Institute, Academy of Sciences, Ondrejov, Czech Republic
[4] VU University Amsterdam, Faculty of Sciences, Department of Mathematics, Amsterdam, The Netherlands

email: mg.schmidt@tum.de

Kolloquium der Leibniz-Sozietät
zum Thema „Geodäsie – Mathematik – Physik – Geophysik“
aus Anlass des 75. Geburtstages von Prof. Dr.-Ing. habil. Dr. h.c. mult. Erik W. Grafarend

Introduction

- Traditionally, **spherical harmonics** (SH) are used for modelling the Earth's gravitational field **globally**.
- These global basis functions are well suited when the generating body has a near **spherical surface**.
- SHs allow **easy computation** of the gravitational potential and its functionals.
- However, measured terrestrial and airborne gravity data are usually reduced to a rotating **reference ellipsoid**.
- Consequently, in such cases **ellipsoidal harmonics** (EH) are more appropriate than SHs.
- Since ellipsoidal basis functions include the geometry of the ellipsoid, the **approximation error** is expected to be less than for spherical basis functions.

Representation in Spherical Harmonics

$$V(r,\lambda,\theta) = \frac{GM}{R} \sum_{n,m} \left(\frac{R}{r}\right)^{n+1} \left(\overline{C}^s_{n,m} \cos(m\lambda) + \overline{S}^s_{n,m} \sin(m\lambda)\right) \overline{P}_{n,m}(\cos\theta)$$

GM	=	gravitational constant
R	=	semi-major axis of the reference ellipsoid (or radius of the bounding sphere)
r, λ, θ	=	radial distance, spherical longitude, spherical co-latitude
n, m	=	degree, order of the expansion
$\overline{P}_{n,m}$	=	**fully normalized associated Legendre function of the 1st kind**

Representation in Spherical Harmonics

$$V(r,\lambda,\theta) = \frac{GM}{R} \sum_{n,m} \left(\frac{R}{r}\right)^{n+1} \left(\overline{C}^s_{n,m} \cos(m\lambda) + \overline{S}^s_{n,m} \sin(m\lambda)\right) \overline{P}_{n,m}(\cos\theta)$$

GM	=	gravitational constant
R	=	semi-major axis of the reference ellipsoid (or radius of the bounding sphere)
r, λ, θ	=	radial distance, spherical longitude, spherical co-latitude
n, m	=	degree, order of the expansion
$\overline{P}_{n,m}$	=	**fully normalized associated Legendre function of the 1st kind**

- The ratio of the **geocentric parameters R and r** defines the **upward continuation** property.

Representation in Spherical Harmonics

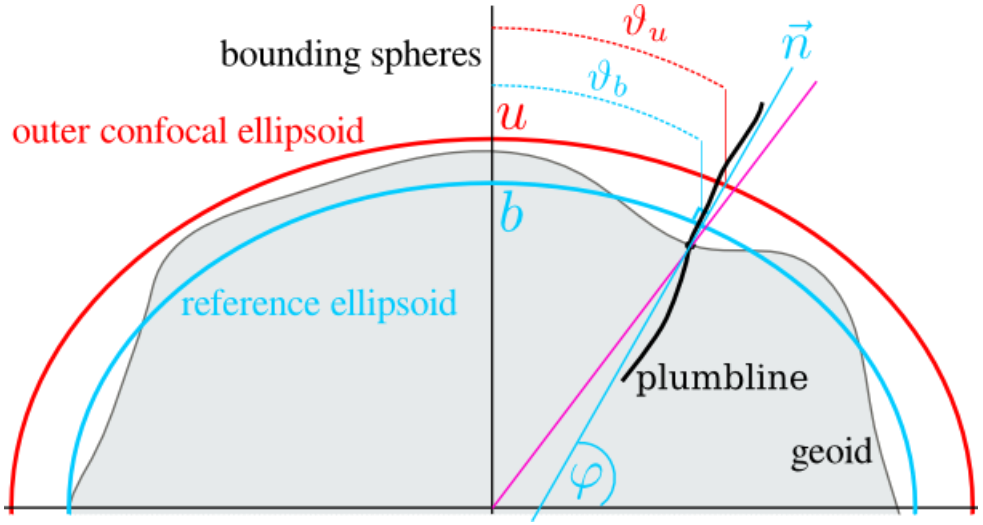
$$V(r,\lambda,\theta) = \frac{GM}{R} \sum_{n,m} \left(\frac{R}{r}\right)^{n+1} \left(\overline{C}^{s}_{n,m} \cos(m\lambda) + \overline{S}^{s}_{n,m} \sin(m\lambda)\right) \overline{P}_{n,m}(\cos\theta)$$

GM	=	gravitational constant
R	=	semi-major axis of the reference ellipsoid (or radius of the bounding sphere)
r, λ, θ	=	radial distance, spherical longitude, spherical co-latitude
n, m	=	degree, order of the expansion
$\overline{P}_{n,m}$	=	**fully normalized associated Legendre function of the 1st kind**

- The ratio of the **geocentric parameters R and r** defines the **upward continuation** property.
- The harmonic coefficients $\bar{C}^{s}_{n,m}$ and $\bar{S}^{s}_{n,m}$ are the **spherical harmonic coefficients**.

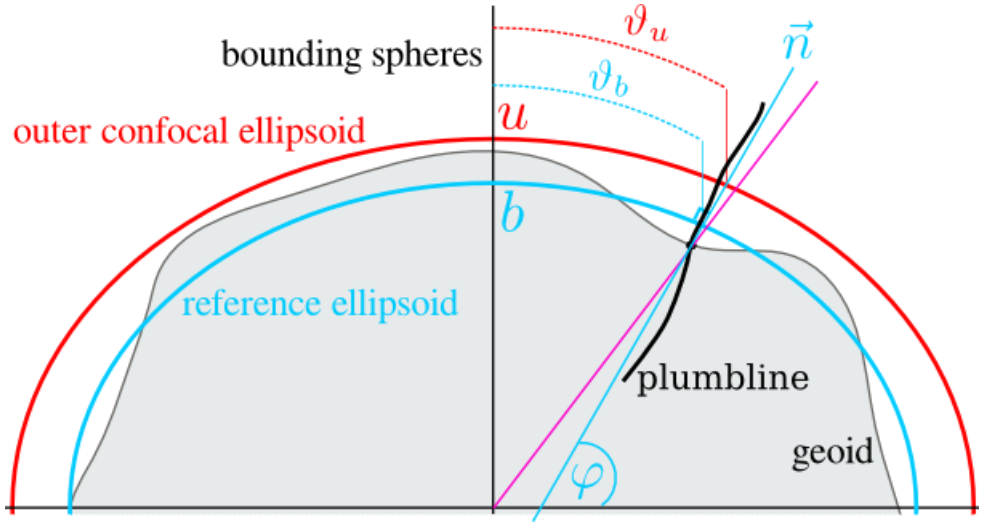
Motivation for Ellipsoidal Modelling

- To get basis functions **closer** to a gravity measurement
- **Less prior corrections** or iterations needed



Motivation for Ellipsoidal Modelling

- To get basis functions **closer** to a gravity measurement
- **Less prior corrections** or iterations needed



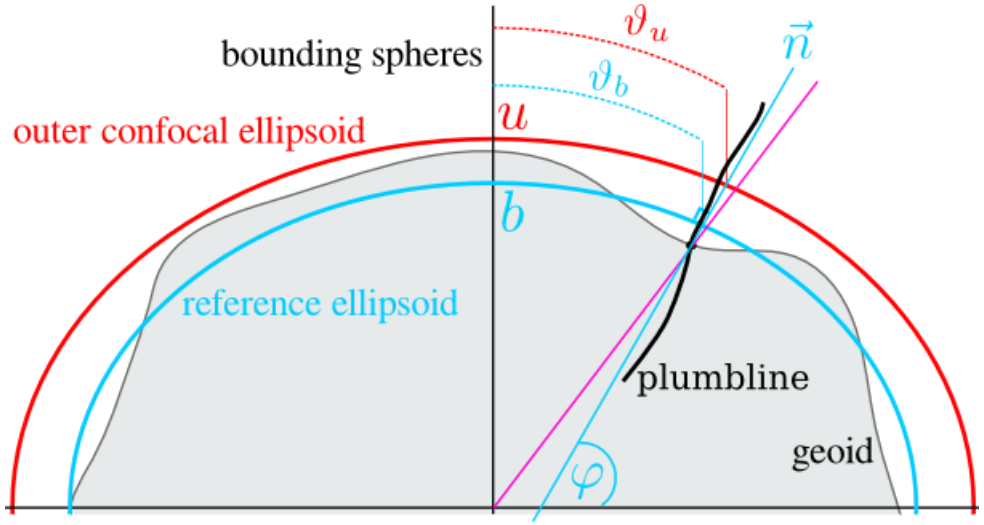
$$
\begin{aligned}
u &= \text{semi-minor axis of confocal ellipsoids} \\
\lambda &= \text{spheroidal longitude} \\
\vartheta &= \text{reduced (spheroidal) co-latitude}
\end{aligned}
$$

$$
\begin{aligned}
x &= \sqrt{u^2 + E^2}\,\sin\vartheta\,\cos\lambda = r\,\sin\theta\,\cos\lambda \\
y &= \sqrt{u^2 + E^2}\,\sin\vartheta\,\sin\lambda = r\,\sin\theta\,\sin\lambda \\
z &= u\cos\vartheta = r\,\cos\theta
\end{aligned}
$$

$$E = \sqrt{a^2 - b^2}$$

Motivation for Ellipsoidal Modelling

- To get basis functions **closer** to a gravity measurement
- **Less prior corrections** or iterations needed

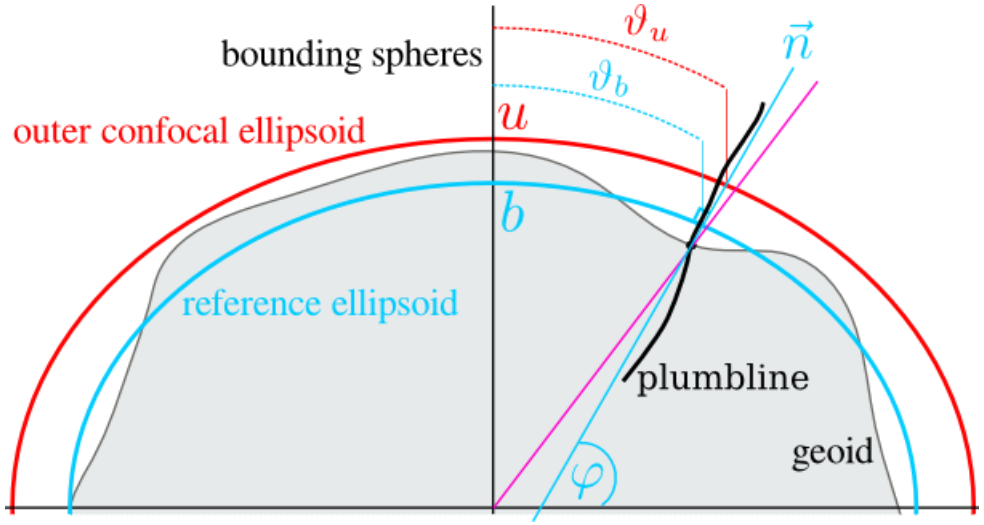


In general, BVPs are ellipsoidal, if one or both requirements hold:

- **Orientation**: a reference direction is the ellipsoidal normal (to approximate plumbline's tangent)
- **Position**: gravity data are reduced to the ellipsoid

Motivation for Ellipsoidal Modelling

- To get basis functions **closer** to a gravity measurement
- **Less prior corrections** or iterations needed



In general, BVPs are ellipsoidal, if one or both requirements hold:

- **Orientation**: a reference direction is the ellipsoidal normal (to approximate plumbline's tangent)
- **Position**: gravity data are reduced to the ellipsoid

Note, there are not many solutions that would solve the problem with both settings. For example, EGM2008 has used data on the ellipsoid, but the data were corrected to the spherical normal before the analysis (Pavlis 2012, JGR).

Representation in Ellipsoidal Harmonics

$$V(u,\lambda,\vartheta) = \frac{GM}{a} \sum_{n,m} \frac{Q_{n,m}(\frac{u}{E})}{Q_{n,m}(\frac{b}{E})} \left(\overline{C}^{e}_{n,m} \cos(m\lambda) + \overline{S}^{e}_{n,m} \sin(m\lambda)\right) \overline{P}_{n,m}(\cos\vartheta)$$

GM	=	gravitational constant
a	=	semi-major axis of the reference ellipsoid
u, λ, ϑ	=	semi-minor axis, spheroidal longitude, spheroidal co-latitude
n, m	=	degree, order of the expansion
$\overline{P}_{n,m}$	=	**fully normalized associated Legendre function of the 1st kind**
$Q_{n,m}$	=	**associated Legendre function of the 2nd kind**

Representation in Ellipsoidal Harmonics

$$V(u,\lambda,\vartheta) = \frac{GM}{a} \sum_{n,m} \frac{Q_{n,m}(\frac{u}{E})}{Q_{n,m}(\frac{b}{E})} \left(\overline{C}^{e}_{n,m} \cos(m\lambda) + \overline{S}^{e}_{n,m} \sin(m\lambda)\right) \overline{P}_{n,m}(\cos\vartheta)$$

GM	=	gravitational constant
a	=	semi-major axis of the reference ellipsoid
u, λ, ϑ	=	semi-minor axis, spheroidal longitude, spheroidal co-latitude
n, m	=	degree, order of the expansion
$\overline{P}_{n,m}$	=	**fully normalized associated Legendre function of the 1st kind**
$Q_{n,m}$	=	**associated Legendre function of the 2nd kind**

- The ratio of the **associated Legendre functions of the 2nd kind** defines the **upward continuation** property.

Representation in Ellipsoidal Harmonics

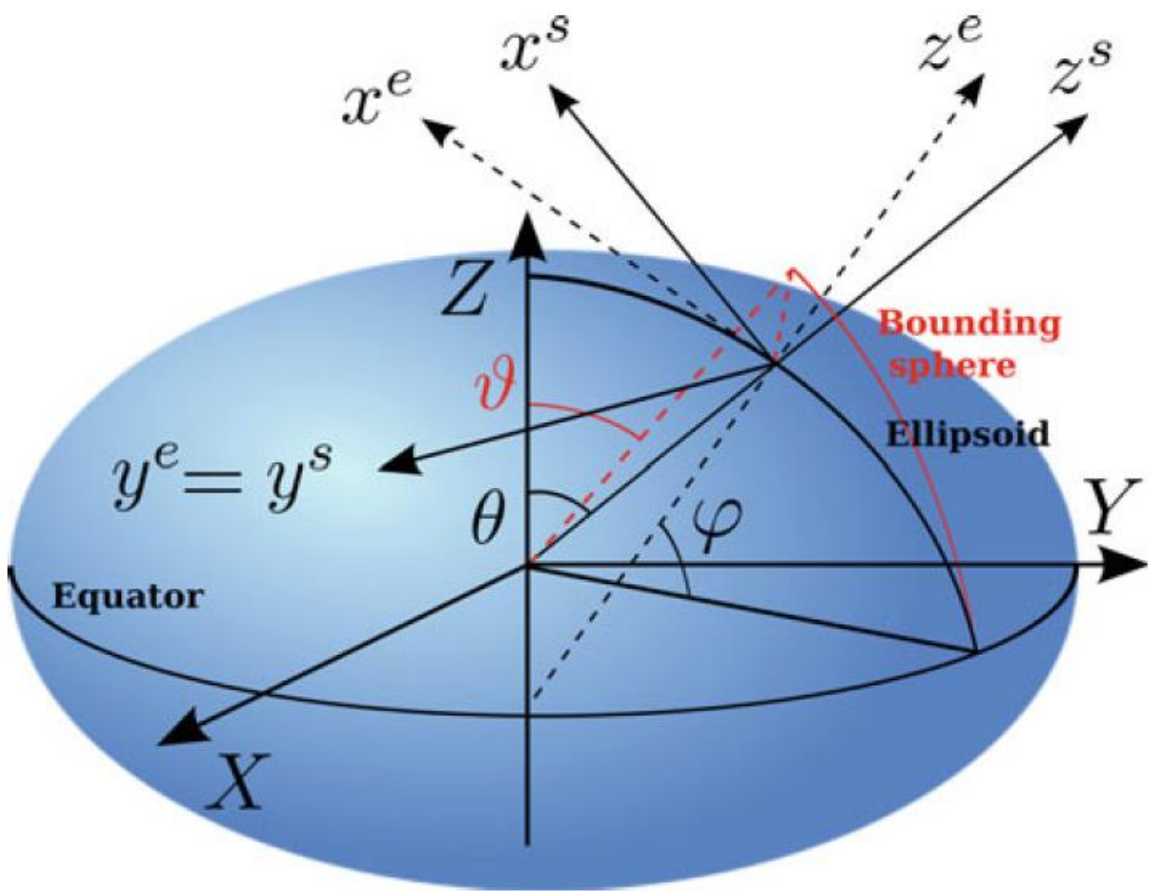
$$V(u,\lambda,\vartheta) = \frac{GM}{a} \sum_{n,m} \frac{Q_{n,m}(\frac{u}{E})}{Q_{n,m}(\frac{b}{E})} \left(\overline{C}^{e}_{n,m} \cos(m\lambda) + \overline{S}^{e}_{n,m} \sin(m\lambda)\right) \overline{P}_{n,m}(\cos\vartheta)$$

GM	=	gravitational constant
a	=	semi-major axis of the reference ellipsoid
u, λ, ϑ	=	semi-minor axis, spheroidal longitude, spheroidal co-latitude
n, m	=	degree, order of the expansion
$\overline{P}_{n,m}$	=	**fully normalized associated Legendre function of the 1st kind**
$Q_{n,m}$	=	**associated Legendre function of the 2nd kind**

- The ratio of the **associated Legendre functions of the 2nd kind** defines the **upward continuation** property.
- The harmonic coefficients $\bar{C}^{e}_{n,m}$ and $\bar{S}^{e}_{n,m}$ are the **ellipsoidal harmonic coefficients**.

Representation in Ellipsoidal Harmonics

$$V(u,\lambda,\vartheta) = \frac{GM}{a} \sum_{n,m} \frac{Q_{n,m}(\frac{u}{E})}{Q_{n,m}(\frac{b}{E})} \left(\overline{C}^{e}_{n,m} \cos(m\lambda) + \overline{S}^{e}_{n,m} \sin(m\lambda)\right) \overline{P}_{n,m}(\cos\vartheta)$$



Definition of **Local North Oriented Frames**

- $\mathrm{LNOF}^{\mathrm{s}}$,
- $\mathrm{LNOF}^{\mathrm{e}}$;

taken from Sebera et al. (2012)

Representation in Ellipsoidal Harmonics

$$V(u,\lambda,\vartheta) = \frac{GM}{a} \sum_{n,m} \frac{Q_{n,m}(\frac{u}{E})}{Q_{n,m}(\frac{b}{E})} \left(\overline{C}^e_{n,m} \cos(m\lambda) + \overline{S}^e_{n,m} \sin(m\lambda)\right) \overline{P}_{n,m}(\cos\vartheta)$$

Spherical	Ellipsoidal
$V_x = -\frac{1}{r} V_\theta$	$-\frac{1}{L} V_\vartheta$
$V_y = -\frac{1}{r \sin\theta} V_\lambda$	$-\frac{1}{v \sin\vartheta} V_\lambda$
$V_z = V_r$	$\frac{v}{L} V_u$
$V_{xx} = \frac{1}{r} V_r + \frac{1}{r^2} V_{\theta\theta}$	$\frac{uv^2}{L^4} V_u + \frac{1}{L^2} V_{\vartheta\vartheta} + \frac{E^2 \cos\vartheta \sin\vartheta}{L^4} V_\vartheta$
$V_{yy} = \frac{1}{r} V_r + \frac{1}{r^2 \sin^2\theta} V_{\lambda\lambda} + \frac{\coth\theta}{r^2} V_\theta$	$\frac{u}{L^2} V_u + \frac{1}{v^2 \sin^2\vartheta} V_{\lambda\lambda} + \frac{\cot\vartheta}{L^2} V_\vartheta$
$V_{zz} = V_{rr}$	$\frac{v^2}{L^2} V_{uu} - \frac{uE^2 \sin^2\vartheta}{L^4} V_u - \frac{E^2 \cos\vartheta \sin\vartheta}{L^4} V_\vartheta$
$V_{xy} = \frac{1}{r^2 \sin\theta} V_{\theta\lambda} - \frac{\cos\theta}{r^2 \sin^2\theta} V_\lambda$	$\frac{1}{vL \sin\vartheta} V_{\vartheta\lambda} - \frac{\cot\vartheta}{vL \sin\vartheta} V_\lambda$
$V_{xz} = \frac{1}{r^2} V_\theta - \frac{1}{r} V_{r\theta}$	$\frac{uv}{L^4} V_\vartheta - \frac{v}{L^2} V_{u\vartheta} - \frac{vE^2 \cos\vartheta \sin\vartheta}{L^4} V_u$
$V_{yz} = \frac{1}{r^2 \sin\theta} V_\lambda - \frac{1}{r \sin\theta} V_{r\lambda}$	$\frac{u}{Lv^2 \sin\vartheta} V_\lambda + \frac{1}{L \sin\vartheta} V_{u\lambda}$

Cartesian derivatives in the LNOFs; taken from R. Koop (1993)

Multi-Resolution Representation (MRR)

- The representation of the gravity field in terms of spherical or ellipsoidal harmonics is only appropriate if the input data is
  - **distributed homogeneously** over the globe and of
  - **similar accuracy**.
- Gravity field modelling in terms of **spherical (radial) basis functions** means an appropriate alternative and is nowadays successfully used in **regional** applications.
- The application of the **wavelet transform** allows the decomposition of a signal F into a certain number of **detail signals** G_j each related to a specific **frequency band** (resolution level $j \in \mathbb{N}_0$).
- This decomposition means a **multi-resolution representation** (MRR).
- Here we discuss the MRR briefly in the context of **ellipsoidal modelling**; for details see, e.g., Schmidt and Fabert (2008)

MRR: Example

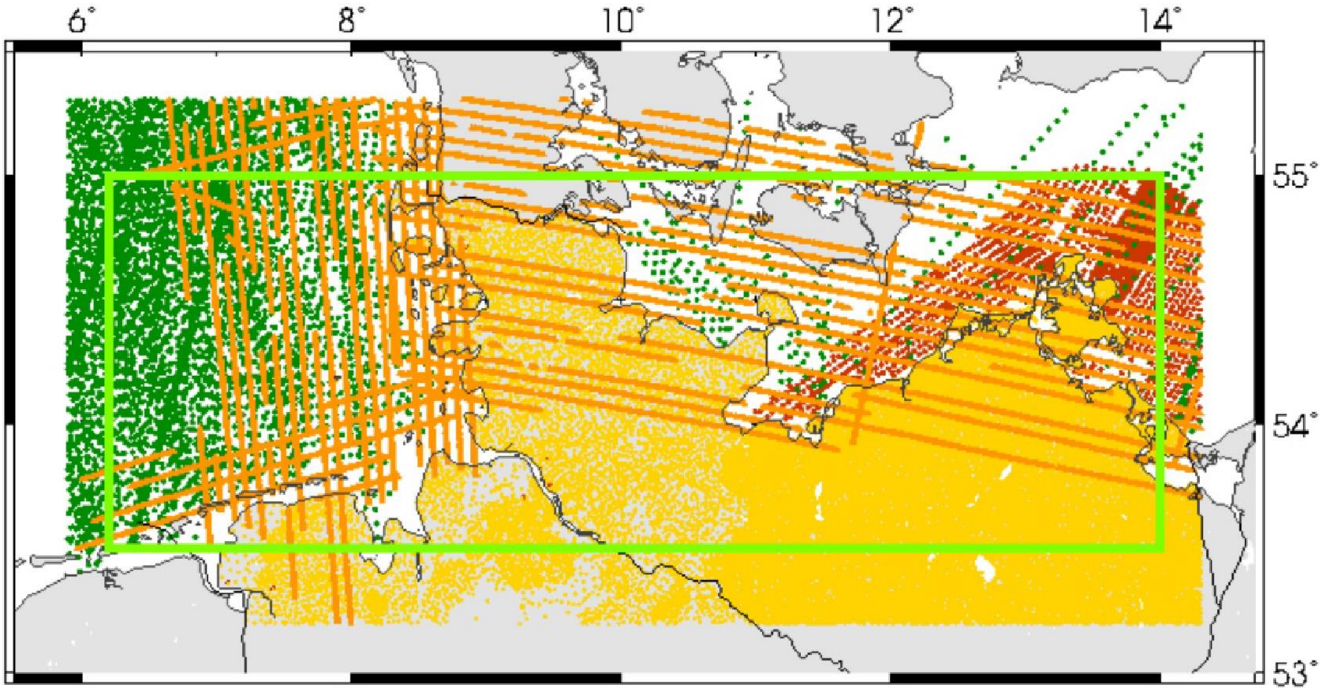
Data sets:

- altimetry,
- airborne-,
- bathymetry-,
- terrestrial measurements

(data kindly provided by BKG)

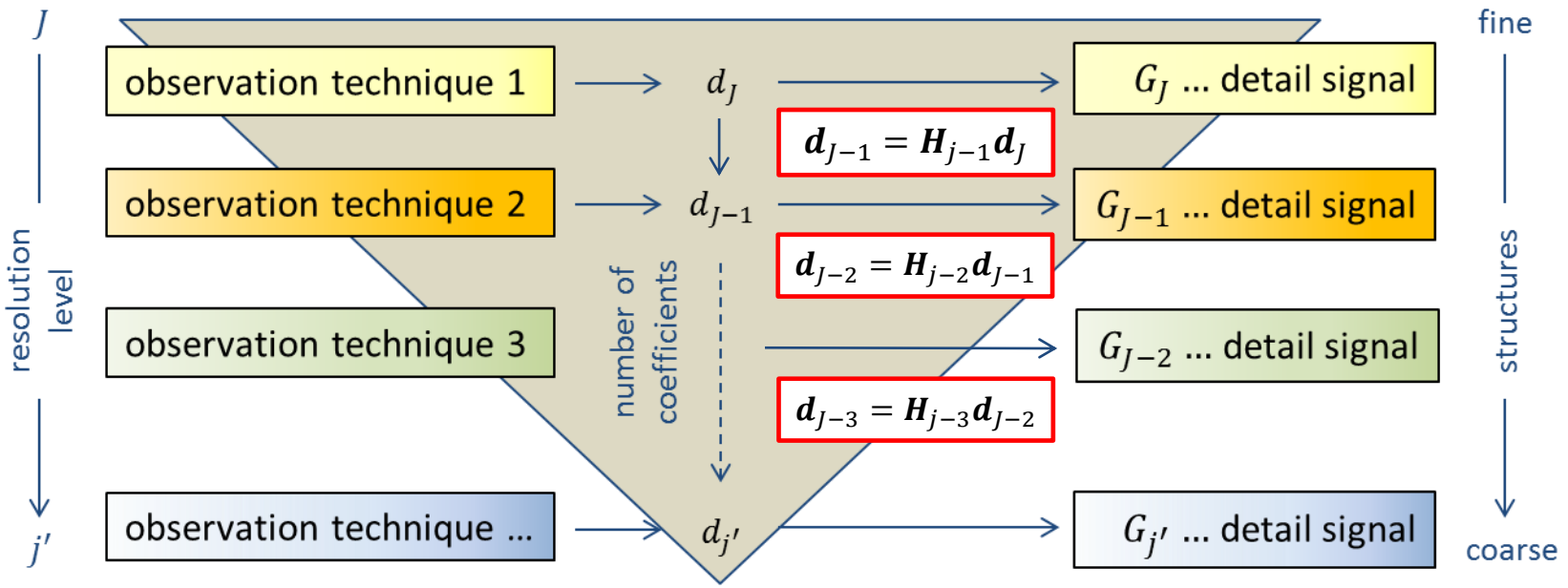
Additional data:

+ GOCE GG
+ GRACE ΔV

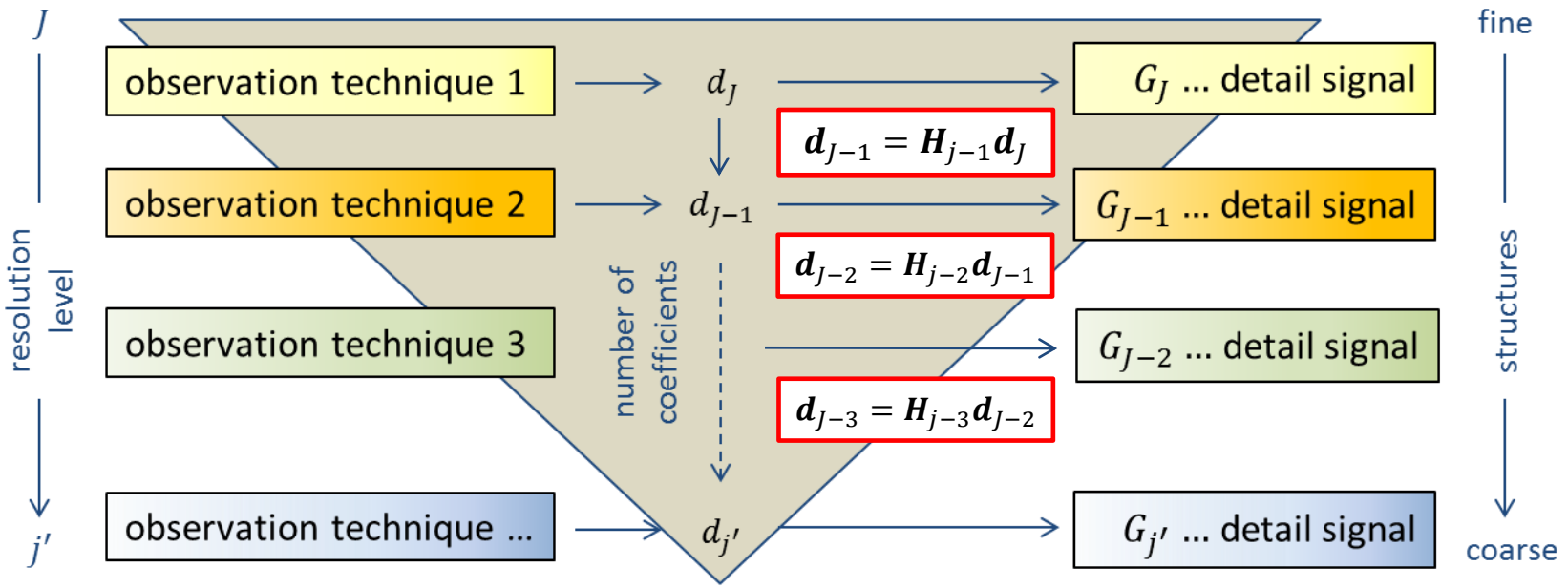


- The **different data sets**
  - ➢ are sensitive to specific **frequency bands** and
  - ➢ have a **spatial resolution** depending on the measurement campaign.
- Thus, they are related to well defined **resolution levels** within the MRR.

Pyramid Algorithm

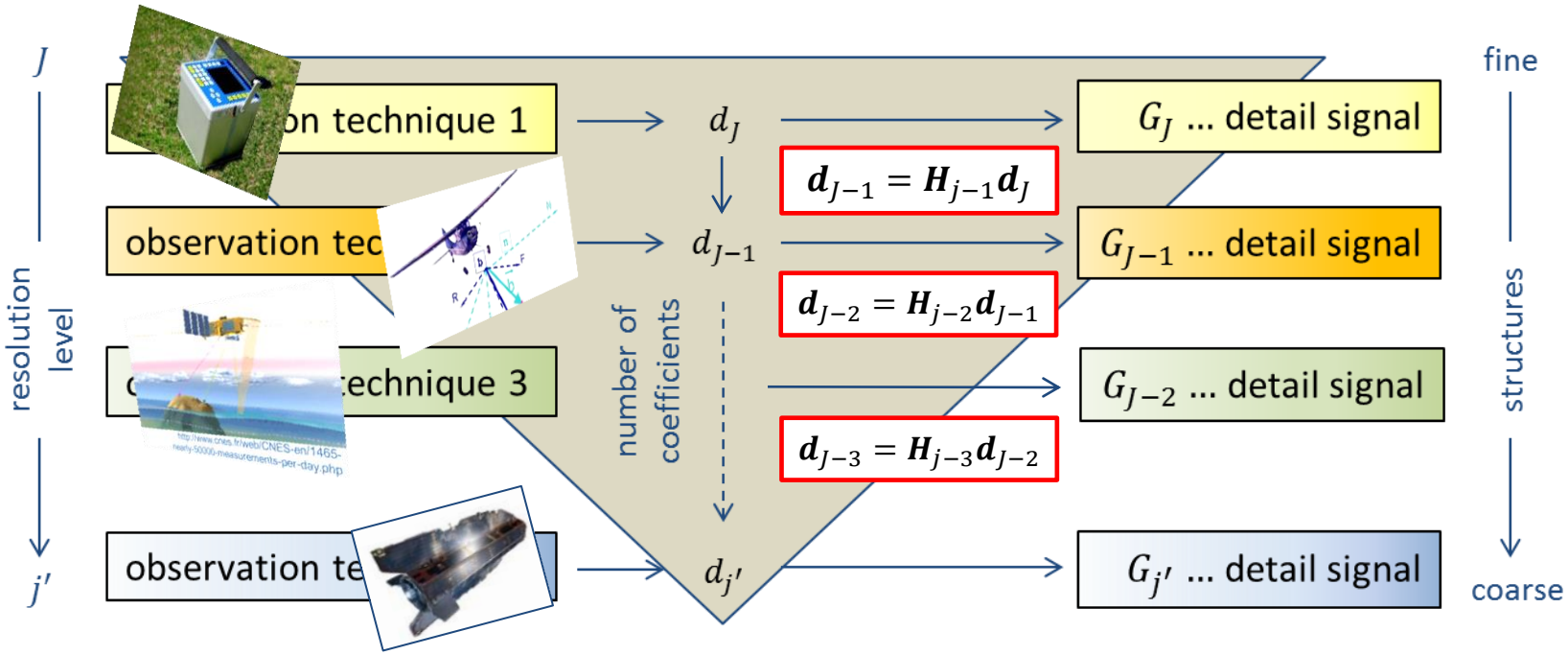


Pyramid Algorithm



Different detail signals $G_J, G_{J-1}, \ldots, G_{j'}$ can be computed from different observation techniques considering the low-pass filter matrices $\boldsymbol{H}_{J-1}, \ldots$

Pyramid Algorithm



Different detail signals $G_J, G_{J-1}, \ldots, G_{j'}$ can be computed from different observation techniques considering the low-pass filter matrices $\boldsymbol{H}_{J-1}, \ldots$

Multi-Resolution Representation (MRR)

- Formulation of the **MRR** for $F(u,\lambda,\vartheta) := V(u,\lambda,\vartheta) - V_{\text{back}}(u,\lambda,\vartheta)$ or **functionals** such as gravity anomalies, gravity gradients, etc.:

$$F(u,\lambda,\vartheta) \approx F_{J+1}(u,\lambda,\vartheta) = F_{j'}(u,\lambda,\vartheta) + \sum_{j=j'}^{J} G_j(u,\lambda,\vartheta)$$

$$F_{j'}(u,\lambda,\vartheta) = \text{low-pass filtered signal}$$

$$G_j(u,\lambda,\vartheta) = \text{band-pass filtered detail signal}$$

Multi-Resolution Representation (MRR)

- Formulation of the **MRR** for $F(u,\lambda,\vartheta) := V(u,\lambda,\vartheta) - V_{\text{back}}(u,\lambda,\vartheta)$ or **functionals** such as gravity anomalies, gravity gradients, etc.:

$$F(u,\lambda,\vartheta) \approx F_{J+1}(u,\lambda,\vartheta) = F_{j'}(u,\lambda,\vartheta) + \sum_{j=j'}^{J} G_j(u,\lambda,\vartheta)$$

$$F_{j'}(u,\lambda,\vartheta) = \text{low-pass filtered signal}$$

$$G_j(u,\lambda,\vartheta) = \text{band-pass filtered detail signal}$$

$$\psi_j(P,P') = \sum_{n=0}^{N_j} \sum_{m=-n}^{n} \frac{Q_{n,|m|}(\frac{u}{E})}{Q_{n,|m|}(\frac{b}{E})} \Psi_{j;n,m}\, Y_{n,m}(P')\, Y_{n,m}(P)$$

Multi-Resolution Representation (MRR)

- Formulation of the **MRR** for $F(u,\lambda,\vartheta) := V(u,\lambda,\vartheta) - V_{\text{back}}(u,\lambda,\vartheta)$ or **functionals** such as gravity anomalies, gravity gradients, etc.:

$$F(u,\lambda,\vartheta) \approx F_{J+1}(u,\lambda,\vartheta) = F_{j'}(u,\lambda,\vartheta) + \sum_{j=j'}^{J} G_j(u,\lambda,\vartheta)$$

- The Legendre coefficients $\Psi_{j;n,m}$ define the shape of the **level-*j* wavelet function** $\psi_j(P,P')$ depending on the two points $P(u,\lambda,\vartheta)$ and $P'(b,\lambda',\vartheta')$.

$$\psi_j(P,P') = \sum_{n=0}^{N_j} \sum_{m=-n}^{n} \frac{Q_{n,|m|}(\frac{u}{E})}{Q_{n,|m|}(\frac{b}{E})} \Psi_{j;n,m} Y_{n,m}(P')\, Y_{n,m}(P)$$

Multi-Resolution Representation (MRR)

- Formulation of the **MRR** for $F(u,\lambda,\vartheta) := V(u,\lambda,\vartheta) - V_{\text{back}}(u,\lambda,\vartheta)$ or **functionals** such as gravity anomalies, gravity gradients, etc.:

$$F(u,\lambda,\vartheta) \approx F_{J+1}(u,\lambda,\vartheta) = F_{j'}(u,\lambda,\vartheta) + \sum_{j=j'}^{J} G_j(u,\lambda,\vartheta)$$

- The Legendre coefficients $\Psi_{j;n,m}$ define the shape of the **level-*j* wavelet function** $\psi_j(P,P')$ depending on the two points $P(u,\lambda,\vartheta)$ and $P'(b,\lambda',\vartheta')$.
- For $\Psi_{j;n,m} = 1$ and $N_j \to \infty$ we get the **Poisson kernel.**

$$\psi_j(P,P') = \sum_{n=0}^{N_j} \sum_{m=-n}^{n} \frac{Q_{n,|m|}\left(\frac{u}{E}\right)}{Q_{n,|m|}\left(\frac{b}{E}\right)} \Psi_{j;n,m} Y_{n,m}(P')\, Y_{n,m}(P)$$

Multi-Resolution Representation (MRR)

- Formulation of the **MRR** for $F(u,\lambda,\vartheta) := V(u,\lambda,\vartheta) - V_{\text{back}}(u,\lambda,\vartheta)$ or **functionals** such as gravity anomalies, gravity gradients, etc.:

$$F(u,\lambda,\vartheta) \approx F_{J+1}(u,\lambda,\vartheta) = F_{j'}(u,\lambda,\vartheta) + \sum_{j=j'}^{J} G_j(u,\lambda,\vartheta)$$

- The level-*j* wavelet function $\psi_j(P,P')$ depends on the ratio of the **associated Legendre functions of 2nd kind (ALF 2)**.

$$\psi_j(P,P') = \sum_{n=0}^{N_j} \sum_{m=-n}^{n} \frac{Q_{n,|m|}(\frac{u}{E})}{Q_{n,|m|}(\frac{b}{E})} \Psi_{j;n,m}\, Y_{n,m}(P')\, Y_{n,m}(P)$$

Multi-Resolution Representation (MRR)

- Formulation of the **MRR** for $F(u,\lambda,\vartheta) := V(u,\lambda,\vartheta) - V_{\text{back}}(u,\lambda,\vartheta)$ or **functionals** such as gravity anomalies, gravity gradients, etc.:

$$F(u,\lambda,\vartheta) \approx F_{J+1}(u,\lambda,\vartheta) = F_{j'}(u,\lambda,\vartheta) + \sum_{j=j'}^{J} G_j(u,\lambda,\vartheta)$$

- The level-*j* wavelet function $\psi_j(P,P')$ depends on the ratio of the **associated Legendre functions of 2nd kind (ALF 2)**.
- For **terrestrial measurements** with a resolution of 2′ we setup the highest resolution level to $J = 12$, thus the **maximum degree value** is $N_J = 4095$.

$$\psi_j(P,P') = \sum_{n=0}^{N_j} \sum_{m=-n}^{n} \frac{Q_{n,|m|}\left(\frac{u}{E}\right)}{Q_{n,|m|}\left(\frac{b}{E}\right)} \Psi_{j;n,m}\, Y_{n,m}(P')\, Y_{n,m}(P)$$

Associated Legendre Functions of 2nd kind

- For the computation of the ALF 2 we use the **hypergeometric formulation** with Jekeli's renormalization.
- Thus, the ALFs 2 are relaced by **Jekeli's function**

$$\overline{S}^0_{n,m}\left(\frac{u}{E}\right) = \left(1+\frac{E^2}{u^2}\right)^{\frac{m}{2}} \left(\frac{R}{u}\right)^{n+1} {}_2F_1\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}, n+\frac{3}{2}, -\frac{E^2}{u^2}\right)$$

➢ can be computed in a straightforward recurrence (Jekeli, 1988)

Associated Legendre Functions of 2nd kind

$$V(u,\lambda,\vartheta) = \frac{GM}{a} \sum_{n,m} \frac{\overline{S}^0_{n,m}(\frac{u}{E})}{\overline{S}^0_{n,m}(\frac{b}{E})} \left(\overline{C}^e_{n,m} \cos(m\lambda) + \overline{S}^e_{n,m} \sin(m\lambda)\right) \overline{P}_{n,m}(\cos\vartheta)$$

$$\overline{S}^0_{n,m}\left(\frac{u}{E}\right) = \left(1 + \frac{E^2}{u^2}\right)^{\frac{m}{2}} \left(\frac{R}{u}\right)^{n+1} {}_2F_1\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}, n+\frac{3}{2}, -\frac{E^2}{u^2}\right)$$

- can be computed in a straightforward recurrence (Jekeli, 1988)

Associated Legendre Functions of 2nd kind

$$V(u,\lambda,\vartheta) = \frac{GM}{a} \sum_{n,m} \frac{\overline{S}^0_{n,m}(\frac{u}{E})}{\overline{S}^0_{n,m}(\frac{b}{E})} \left(\overline{C}^e_{n,m} \cos(m\lambda) + \overline{S}^e_{n,m} \sin(m\lambda)\right) \overline{P}_{n,m}(\cos\vartheta)$$

$$\overline{S}^0_{n,m}\left(\frac{u}{E}\right) = \left(1+\frac{E^2}{u^2}\right)^{\frac{m}{2}} \left(\frac{R}{u}\right)^{n+1} {}_2F_1\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}, n+\frac{3}{2}, -\frac{E^2}{u^2}\right)$$

- can be computed in a straightforward recurrence (Jekeli, 1988)

$${}^*\overline{S}^0_{n,m}\left(\frac{u}{E}\right) = \left(1+\frac{E^2}{u^2}\right)^{-\frac{m}{2}} \left(\frac{R}{u}\right)^{n+1} {}_2F_1\left(\frac{n-m+2}{2}, \frac{n-m+1}{2}, n+\frac{3}{2}, -\frac{E^2}{u^2}\right)$$

- the same function as above but with faster convergence (Sebera et al., 2012)

Associated Legendre Functions of 2nd kind

$$V(u,\lambda,\vartheta) = \frac{GM}{a} \sum_{n,m} \frac{\overline{S}^0_{n,m}(\frac{u}{E})}{\overline{S}^0_{n,m}(\frac{b}{E})} \left(\overline{C}^e_{n,m} \cos(m\lambda) + \overline{S}^e_{n,m} \sin(m\lambda)\right) \overline{P}_{n,m}(\cos\vartheta)$$

$$\overline{S}^0_{n,m}\left(\frac{u}{E}\right) = \left(1+\frac{E^2}{u^2}\right)^{\frac{m}{2}} \left(\frac{R}{u}\right)^{n+1} {}_2F_1\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}, n+\frac{3}{2}, -\frac{E^2}{u^2}\right)$$

- can be computed in a straightforward recurrence (Jekeli, 1988)

$${}^*\overline{S}^0_{n,m}\left(\frac{u}{E}\right) = \left(1+\frac{E^2}{u^2}\right)^{-\frac{m}{2}} \left(\frac{R}{u}\right)^{n+1} {}_2F_1\left(\frac{n-m+2}{2}, \frac{n-m+1}{2}, n+\frac{3}{2}, -\frac{E^2}{u^2}\right)$$

- the same function as above but with faster convergence (Sebera et al., 2012)

$${}^*\overline{S}^0_{n,m}\left(\frac{u}{E}\right) = \left(\frac{R}{\sqrt{u^2+E^2}}\right)^{n+1} {}_2F_1\left(\frac{n+m+1}{2}, \frac{n-m+1}{2}, n+\frac{3}{2}, \frac{E^2}{u^2+E^2}\right)$$

- the same but much better suitable for high-degree computations; $n > 3000$.

Associated Legendre Functions of 2nd kind

$$V(u,\lambda,\vartheta) = \frac{GM}{a} \sum_{n,m} \frac{{}^*\overline{S}^0_{n,m}(\frac{u}{E})}{{}^*\overline{S}^0_{n,m}(\frac{b}{E})} \left(\overline{C}^e_{n,m} \cos(m\lambda) + \overline{S}^e_{n,m} \sin(m\lambda)\right) \overline{P}_{n,m}(\cos\vartheta)$$

$$\overline{S}^0_{n,m}\left(\frac{u}{E}\right) = \left(1+\frac{E^2}{u^2}\right)^{\frac{m}{2}} \left(\frac{R}{u}\right)^{n+1} {}_2F_1\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}, n+\frac{3}{2}, -\frac{E^2}{u^2}\right)$$

- can be computed in a straightforward recurrence (Jekeli, 1988)

$${}^*\overline{S}^0_{n,m}\left(\frac{u}{E}\right) = \left(1+\frac{E^2}{u^2}\right)^{-\frac{m}{2}} \left(\frac{R}{u}\right)^{n+1} {}_2F_1\left(\frac{n-m+2}{2}, \frac{n-m+1}{2}, n+\frac{3}{2}, -\frac{E^2}{u^2}\right)$$

- the same function as above but with faster convergence (Sebera et al., 2012)

$${}^*\overline{S}^0_{n,m}\left(\frac{u}{E}\right) = \left(\frac{R}{\sqrt{u^2+E^2}}\right)^{n+1} {}_2F_1\left(\frac{n+m+1}{2}, \frac{n-m+1}{2}, n+\frac{3}{2}, \frac{E^2}{u^2+E^2}\right)$$

- the same but much better suitable for high-degree computations; $n > 3000$.

Associated Legendre Functions of 2nd kind

$$\psi_j(P,P') = \sum_{n=0}^{N_j} \sum_{m=-n}^{n} \frac{{}^*\overline{S}^0_{n,|m|}\left(\frac{u}{E}\right)}{{}^*\overline{S}^0_{n,|m|}\left(\frac{b}{E}\right)} \Psi_{j;n,m}\, Y_{n,m}(P')\, Y_{n,m}(P)$$

$$\overline{S}^0_{n,m}\left(\frac{u}{E}\right) = \left(1+\frac{E^2}{u^2}\right)^{\frac{m}{2}} \left(\frac{R}{u}\right)^{n+1} {}_2F_1\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}, n+\frac{3}{2}, -\frac{E^2}{u^2}\right)$$

- can be computed in a straightforward recurrence (Jekeli, 1988)

$${}^*\overline{S}^0_{n,m}\left(\frac{u}{E}\right) = \left(1+\frac{E^2}{u^2}\right)^{-\frac{m}{2}} \left(\frac{R}{u}\right)^{n+1} {}_2F_1\left(\frac{n-m+2}{2}, \frac{n-m+1}{2}, n+\frac{3}{2}, -\frac{E^2}{u^2}\right)$$

- the same function as above but with faster convergence (Sebera et al., 2012)

$${}^*\overline{S}^0_{n,m}\left(\frac{u}{E}\right) = \left(\frac{R}{\sqrt{u^2+E^2}}\right)^{n+1} {}_2F_1\left(\frac{n+m+1}{2}, \frac{n-m+1}{2}, n+\frac{3}{2}, \frac{E^2}{u^2+E^2}\right)$$

- the same but much better suitable for high-degree computations; $n > 3000$.

Summary

- From a wide range of approaches for modelling the Earth's gravitational field we presented two **ellipsoidal** ones:
  - a **global** representation based on ellipsoidal harmonics,
  - a **regional** multi-resolution representation based on ellipsoidal basis functions.
- The chosen approach depends on the **distribution**, the **variety** and the **sensitivity** of the input data sets.
- The presented basis functions require the evaluation of the **associated Legendre functions** of the 2nd kind.
- We use the **hypergeometric formulation** and **Jekeli's renormalization** for computing the associated Legendre functions of 2nd kind.
- One advantage of this procedure is that the functions ${}^{*}\overline{S}^{0}_{n,m}$ are also needed for the **transformation** between SH and EH coefficients.

References

- Jekeli C. (1988): The exact transformation between ellipsoidal and spherical expansions. Manuscripta geodaetica, 13:106-113.
- Koop, R. (1993). Global gravity field modelling using satellite gravity gradiometry. PhD thesis, Netherlands geodetic commission, Delft, The Netherlands.
- Pavlis N. K., Holmes S. A., Kenyon S. C., and Factor J. K. (2012): The development and evaluation of the Earth Gravitational Model 2008 (EGM2008). Journal of Geophysical Research (Solid Earth), 117 (B16): 4406.
- Schmidt M., Fabert O. (2008): Ellipsoidal Wavelet Representation of the Gravity Field. OSU report 487, Columbus, Ohio.
- Sebera J., Bouman J., and Bosch W. (2012): On computing ellipsoidal harmonics using Jekelis renormalization. J Geod, 1:14, 10.1007/s00190-012-0549-4.
- Sebera J., Wagner C.A., Bezděk A., and Klokočník, J. (2013): Short guide to direct gravitational field modelling with Hotine's equations. J Geod, 87:223–238, 10.1007/s00190-012-0591-2.
- Sebera J., Bezděk A., Kostelecký J., Pešek I, and Shum C.K. (2015): An ellipsoidal approach to update a high-resolution geopotential model over the oceans: study case using EGM2008 and DTU10 (under review).

Backup slides

Associated Legendre Functions of 2nd kind

- For the computation of the ALFs of the 2nd kind we use the **hypergeometric formulation** with **Jekeli's renormalization**

$$\bar{S}^0_{n,m}\left(\frac{u}{E}\right) = \frac{\left(\frac{R}{E}\right)^{n+1} i^{n+1}(2n+1)!}{2^n n!}\sqrt{\frac{\epsilon_m}{(2n+1)(n-m)!(n+m)!}}\bar{Q}_{n,m}(z)$$

$$\bar{Q}_{n,m}(z) = (-1)^m \frac{2^n n!(n+m)!}{(2n+1)!}\frac{\left(z^2-1\right)^{\frac{m}{2}}}{z^{n+m+1}} \underbrace{{}_2F_1\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}, n+\frac{3}{2}, \frac{1}{z^2}\right)}_{\text{Gauss hypergeometric function}} \underbrace{\sqrt{\frac{(2n+1)(n-m)!}{\epsilon_m(n+m)!}}}_{\text{Normalization}}$$

- This gives **Jekeli's function** as

$$\bar{S}^0_{n,m}\left(\frac{u}{E}\right) = \left(1+\frac{E^2}{u^2}\right)^{\frac{m}{2}}\left(\frac{R}{u}\right)^{n+1} {}_2F_1\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}, n+\frac{3}{2}, -\frac{E^2}{u^2}\right)$$

Representation in Ellipsoidal Harmonics

$$V(u,\lambda,\vartheta) = \frac{GM}{a} \sum_{n,m} \frac{\overline{S}^0_{n,m}(\frac{u}{E})}{\overline{S}^0_{n,m}(\frac{b}{E})} \left(\overline{C}^e_{n,m} \cos(m\lambda) + \overline{S}^e_{n,m} \sin(m\lambda)\right) \overline{P}_{n,m}(\cos\vartheta)$$