Reference Frames & Reference Systems in Geodesy

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Grafarend 75 - Berlin 13/02/2015

Reference Frames & Reference Systems on Geodesy

Conceptually a central problem in Geodesy where we want to estimate coordinates from observables that are independent of the Coordinate System chosen (what theoretical physicists call *gauge invariance*)

The rank defect of the deterministic model

$$\underline{y} = \begin{vmatrix} y_1 \\ \vdots \\ y_M \end{vmatrix} = F(\underline{x}) \qquad \underline{x} = \begin{vmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_N \end{vmatrix}$$

$\uparrow$ Observables

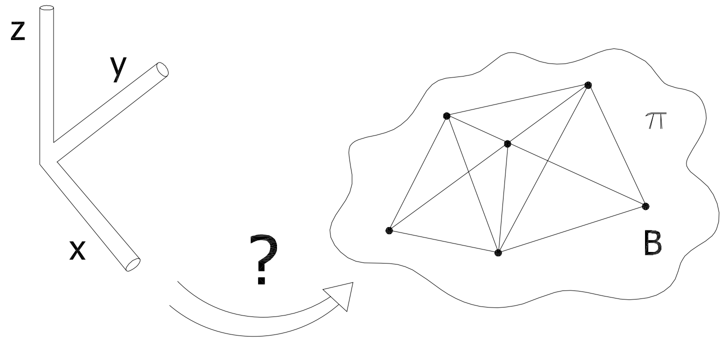
$\uparrow$ Parameters ($3N$ coordinates of N points)

is strictly related to the need of defining a R.S. and a R.F. for further geodetic operations (e.g. networks densification).

Geodetic/Statistical literature on the subject by Bjerhammar, More/Penrose, Rao, Tautenberg, Grafarend...

Classical definition

- A R.S is a physical Cartesian triad in a chosen position with respect to a body B represented by a polyhedron π



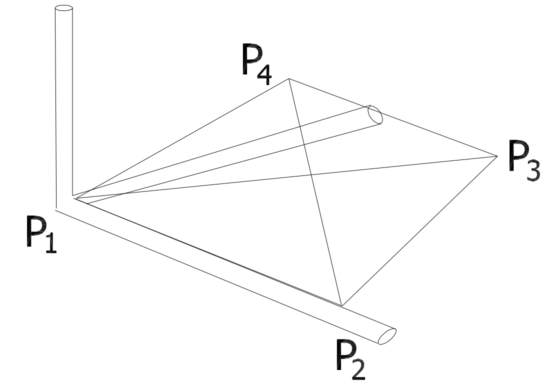
R.S. gives the relative position $(xyz) \Leftrightarrow \pi(\underline{x}_1, \ldots \underline{x}_N)$

- A R.F. is the set of estimated coordinates $(\widehat{\underline{x}}_1, \widehat{\underline{x}}_2, \ldots, \widehat{\underline{x}}_2)$ (the estimated polyhedron $\widehat{\pi}$) in the given R.S. i.e. physical Cartesian triad.

  It is a “frame” in the sense that once $\{\widehat{\underline{x}}_i\}$ are frozen, we can attach to them new points by new observations.

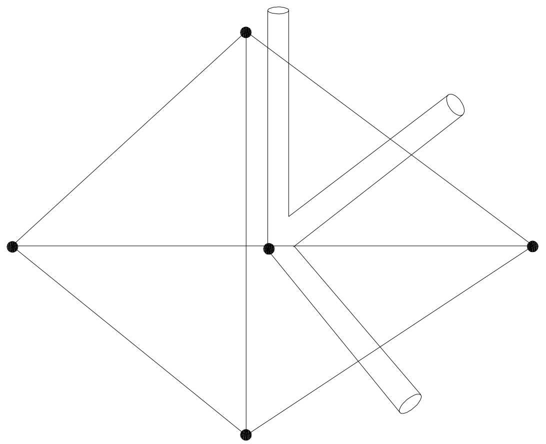
1. Is it correct to try to define first a R.S. and then a R.F.?

No: unless I use 3 points of π to define the R.S.



Classical R.S. attached to π

I need first to know the coordinates of $\underline{x}_i$, i.e. I need to have estimates $\widehat{\pi}(\widehat{\underline{x}})$ to place the coordinate triad



Example: if I want a barycentric triad I need first to know the coordinates of the points

2. Do we really need to define the triad with respect to π or to $\widehat{\pi}$?

No: we could equivalently say that we fix the triad and we try to place π or $\widehat{\pi}$ with respect to it!

Program for a "conceptual suite"

a Define a R.F. i.e. the route from observations Y_0 to $\widehat{\pi}(\underline{\widehat{x}})$

b Define a R.S. in relation to a R.F.
(we need an operative definition according to Einstein)

c Define a quality index of a R.F. based on its "distance" from the R.S. and the optimal R.F

Definition of a R.F.

$$RF \equiv \{\mathcal{O}, \mathcal{M}o, \mathcal{M}e, \mathcal{C}\} \;;$$

- Observations: a vector $\underline{Y}_0$ considered as a sample from an M-dimensional R.V. $\underline{Y}$

  + metadata on the measurement process

- Model: $\mathcal{M}o$-stochastic

$$\underline{Y} = \underline{y} + \underline{\nu}$$

$f_n(\nu)$ known, or at least $E\{\nu\} = 0$, C_ν known

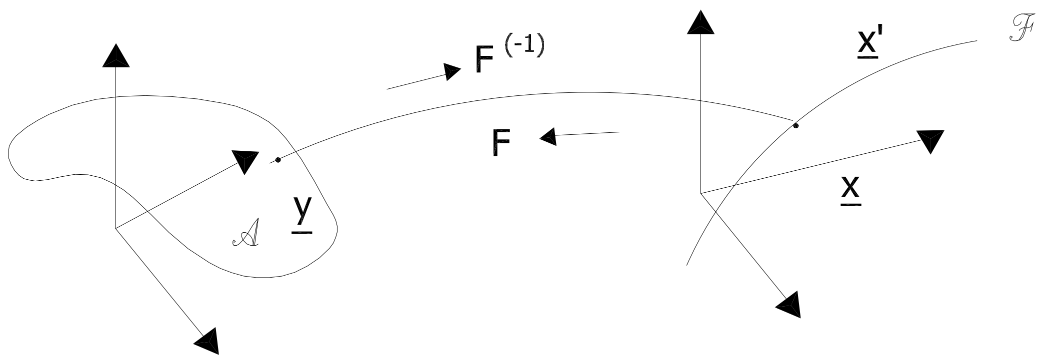
- Model: $\mathcal{M}o$-deterministic

$$\underline{y} = F(\underline{x}) \text{ (we ignore ancillary parameters)}$$

$$\Downarrow$$

- defines the manifold $\mathcal{A}$ of admissible values
- it is rank decificient i.e.

$$\underline{x} \underset{F}{\rightarrow} \underline{y} \text{ (1 value) , } \underline{y} \underset{F^{(-1)}}{\rightarrow} \underline{x} \text{ many values)}$$



The fiber F is defined so that

$$\underline{x}' = G(\underline{x}, \underline{\vartheta}) \quad \underline{\vartheta} 6 \text{ d.f.!}$$

$$\Downarrow$$

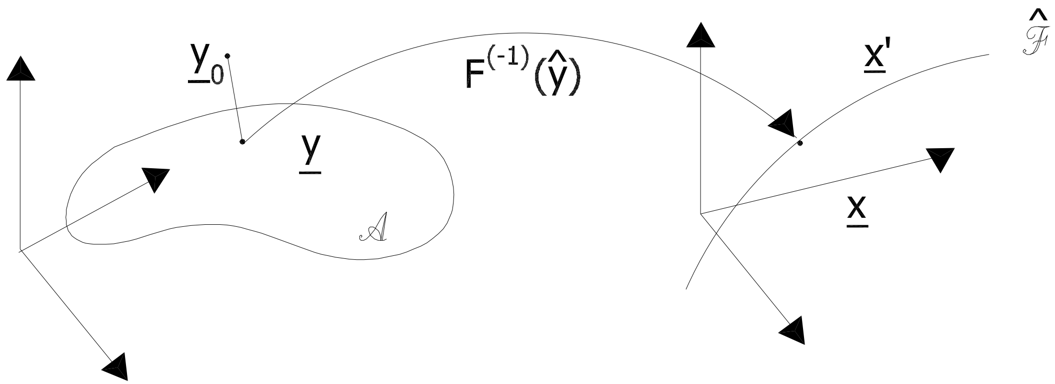
$$F(\underline{x}') \equiv F[G(\underline{x}, \underline{\vartheta})] \equiv G(\underline{x}), \ \forall \underline{\vartheta}$$

Note: $G(\underline{x}, \underline{\vartheta}) = H(\underline{\vartheta})\underline{x}$ rototraslation! known form!

- Method: several possibilities but in Geodesy

$$\mathcal{M}e \equiv \text{ L.S.}$$

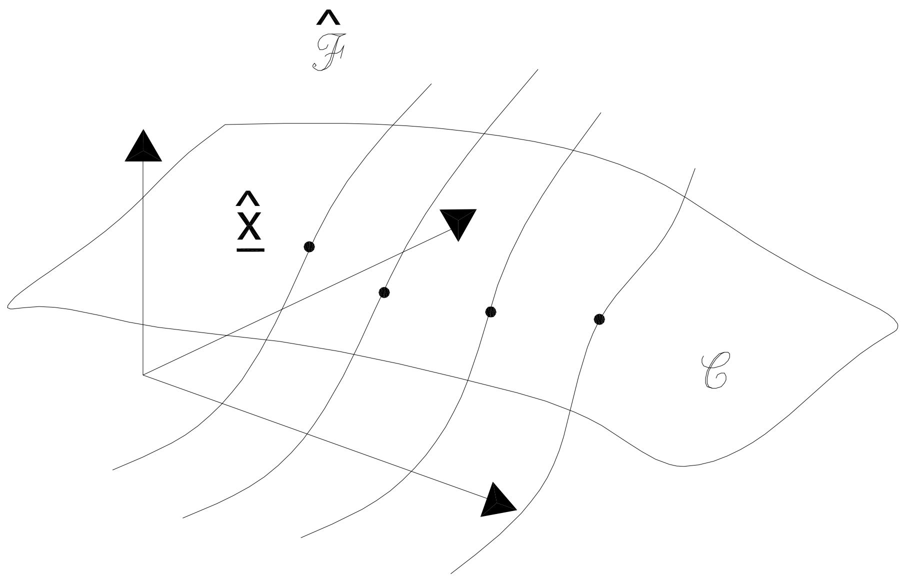
Up to here we have only a path



$\min |Y_0 - \underline{\widehat{y}}|^2_{C_\nu^{-1}}$
$\underline{\widehat{y}} \in \mathcal{A}$

Note: $F^{(-1)}(S)$ is the fiber $\widehat{\mathcal{F}}$ and not a specific $\underline{\widehat{x}}$

- Constraints: $\mathcal{C}$ to go from $\widehat{\mathcal{F}}$ to $\widehat{\underline{x}}$ by crossing all the fibers with a constraining manifold



$$\mathcal{C} \sim R^6$$

$$\mathcal{C}^{comp} \sim R^{n-6} = R^{3N-6}$$

$\mathcal{C} \equiv \{C(\underline{x}) = 0\}$ it can be put into an explicit form by splitting

$$\underline{x} = \begin{vmatrix} \underline{x}_0 \\ \underline{x}_c \end{vmatrix} \quad \begin{matrix} \rightarrow 3N - 6 \\ \rightarrow 6 \text{ suitably chosen coordinates} \end{matrix}$$

$$\underline{x}_c = K(\underline{x}_0)$$

Note: the correspondence

$$\mathcal{A} = \underline{y} = F[\underline{x}_0, K(\underline{x}_0)] \quad \underline{x}_0 \in R^{n-6}$$

is now 1 to 1, i.e. the matrix

$$J = F_{\underline{x}_0} + F_{\underline{x}_c} K_{x_0} \text{ is of full rank}$$

- the choice $K_{\underline{x}_0} \equiv 0$ is always possible
- the matrix $F_{\underline{x}_0}$ alone spans the tangent manifold to $\mathcal{A}$ so $F_{\underline{x}_0}$ is of full rank and

$$F_{\underline{x}_c} = F_{\underline{x}_0} \Gamma \text{ for some } \Gamma \, .$$

i. e.

$$J = F_{\underline{x}_0}(I + \Gamma K_{x_0}) \quad ((I + \Gamma K_{\underline{x}_0}) \text{ regular: transversality condition})$$

So with our definition of R.F. we are able now to solve the L.S. problem

$$\underline{\widehat{x}}_0 = \arg\min | Y_{-0} - F\, [\underline{x}_0,\ K(\underline{x}_0)]|_{C_\nu^{-1}}$$

and compute

$$\underline{\widehat{x}}_c = K(\underline{\widehat{x}}_0)\ .$$

<u>Note</u> that if $\underline{\widehat{x}}_0$ has covariance matrix

$$C_0 = C_{\widehat{x}_0} = (J^T C_\nu^{-1} J)^{-1}$$

then the <u>full</u> vector $\underline{\widehat{x}}$ has covariance matrix

$$C_{\underline{\widehat{x}}} = \begin{vmatrix} C_0 & C_0 K^T_{\underline{x}_0} \\ K_{\underline{x}_0} C_0 & K_{\underline{x}_0} C_0 K^T_{\underline{x}_0} \end{vmatrix} \quad \text{(Note: this is singular)}$$

<u>Note</u>: we could define a reduced $RF \equiv \{\underline{\widehat{x}}\ , C_{\underline{\widehat{x}}}\}$ but never $\underline{\widehat{x}}$ alone!

Definition of R.S.

Assume you have R repetitions $\underline{Y}_{or}$, $r = 1 \ldots R$, of the same observations. Then by using the strong Law of Large Numbers maintaining that

$$\frac{1}{R}\sum_{r=1}^{R}\underline{Y}_{or} \to \underline{y} \in \mathcal{A} \text{ almost surely (a.s.)},$$

we easily prove that for the L.S. solution $\widehat{\underline{x}} = \arg\min \sum_{r=1}^{R} |\underline{Y}_{or} - F(\underline{x})|^2_{C_\nu^{-1}}$ the limit $\lim_{R\to\infty} \widehat{\underline{x}} = \underline{x}$ holds a.s. even for non-linear models. So we can define a R.S. as the body $\pi = \lim_{R\to\infty} \widehat{\pi}$ which attains now a precise position in our coordinate triad!

Note: restriction to linear models/linear constraints.

In this case

$$F(\underline{x}) = A\underline{x} = A_0\underline{x}_0 + A_c\underline{x}_c \qquad A_c = A_0\Gamma$$

$$\underline{x}_c = K(\underline{x}_0) = K\underline{x}_0 \qquad K_{\underline{x}_0} = K$$

$$J = F_{\underline{x}_0} + F_{\underline{x}_c}K_{x_0} = A_0 + A_cK$$

$$C_0 = [(A_0 + A_cK)^T C_\nu^{-1}(A_0 + A_cK)]^{-1}$$

Moreover, in this case L.S. give an unbiased estimate

$$E\{\widehat{\underline{x}}\} = \underline{x}$$

$$(\underline{x}_c = K\underline{x})$$

Quality index

Restricted to linear models, an obvious quality index is

$$\begin{aligned} Q &= E\{|\underline{\widehat{x}} - \underline{x}|^2\} = TrC_{\underline{\widehat{x}}} = \\ &= TrC_0 + TrKC_0K^T = \\ &= TrC_0(I + K^TK) \end{aligned}$$

so we can search for the best R.F. as the one satisfying constraints (i.e. K) such that Q is minimized.

Optimal R.F.: the variational equation is

$$\frac{1}{2}\delta Q = Tr\delta_{K^T}C_0(I + K^T K) + TrC_0\delta K^T K$$

now

$$\delta_{K^T}C_0 = -C_0\delta K^T A_c^T C_\nu^{-1}(A_0 + A_c K)C_0$$

Putting to 0 the first variation we get

$$-A_c^T C_\nu^{-1}(A_0 + A_c K)C_0(I + K^T K)C_0 + KC_0 = 0;$$

even simplifying C_0 from the right a really horrible question!

However

$$A_c = A_0\Gamma \ , \ J = A_0 + A_c K = A_0(I + \Gamma K)$$
$$C_0 = (I + \Gamma K)^{-1}(A_0^T C_\nu^{-1} A_0)^{-1}(I + K^T\Gamma^T)^{-1}$$

Substituing and simplifying we get the notable equation

$$K = \Gamma^T (I + K^T \Gamma^T)^{-1} (I + K^T K)$$

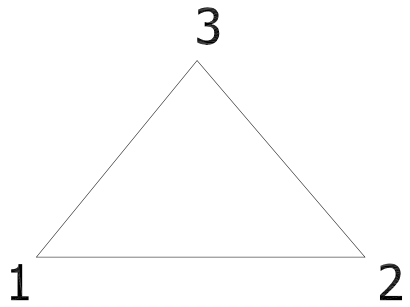
I don't know its general solution, however one solution is certainly

$$\boxed{K = \Gamma^T}$$

Is this the wanted minimum?

I cannot answer but I can show by an elementary example that this is what we are after!

Example



The leveling triangle with

$$\underline{Y} = \begin{vmatrix} Y_1 \\ Y_2 \\ Y_3 \end{vmatrix} = \begin{vmatrix} q_2 - q_1 \\ q_3 - q_2 \\ q_1 - q_3 \end{vmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \underline{x} \; ; \qquad C_\nu = I$$

Put $A_0 = \begin{bmatrix} -1 & 1 \\ 0 & -1 \\ 1 & 0 \end{bmatrix} \quad A_c \begin{vmatrix} 0 \\ 1 \\ -1 \end{vmatrix}$ so that

$$A_c = -A_0 \begin{vmatrix} 1 \\ 1 \end{vmatrix} \text{ i.e. } \Gamma = - \begin{vmatrix} 1 \\ 1 \end{vmatrix} , \Gamma^T = -[1\ 1]$$

But then our optimal frame is

$$x_c = q_3 = \Gamma^T \underline{x}_0 = -(q_1 + q_2) \Rightarrow q_1 + q_2 + q_3 = 0$$

the barycentric frame as everybody would expect!