Applications of fiber bundles in physics: From cosmology to electronic states of matter

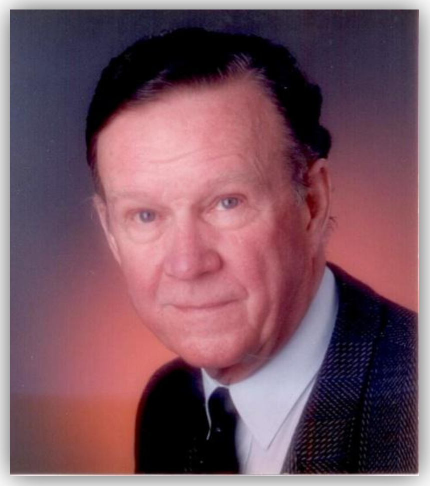
Hans-Rainer Trebin

Institut für Theoretische und Angewandte Physik der Universität Stuttgart, Germany

Colloquium in honor of Prof. Dr. Erik W. Grafarend on occasion of his 75th birthday

Berlin 13 February 2015

Prolog



Erik W. Grafarend and H-RT:

Ekkehart Kröner (1919 – 2000):

- Lehrstuhl für Theoretische Physik, Clausthal (1963 – 1969)
- Lehrstuhl für Theoretische und Angewandte Physik, Stuttgart (1969 – 1985)

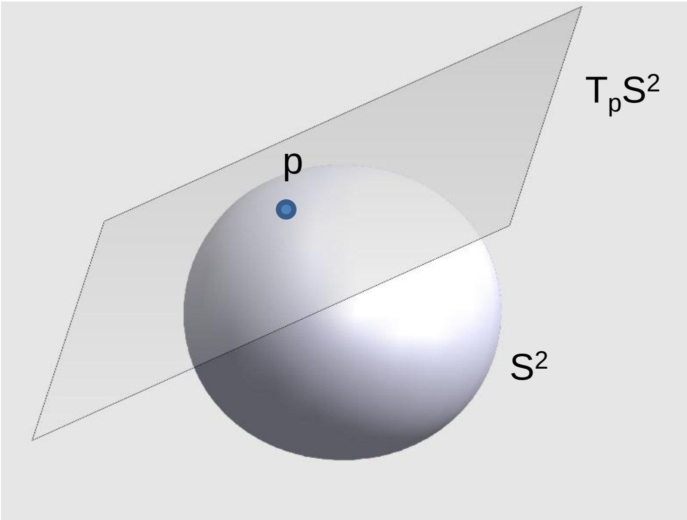
SFB 228 Hochgenaue Navigation – Integration navigatorischer und geodätischer Methoden

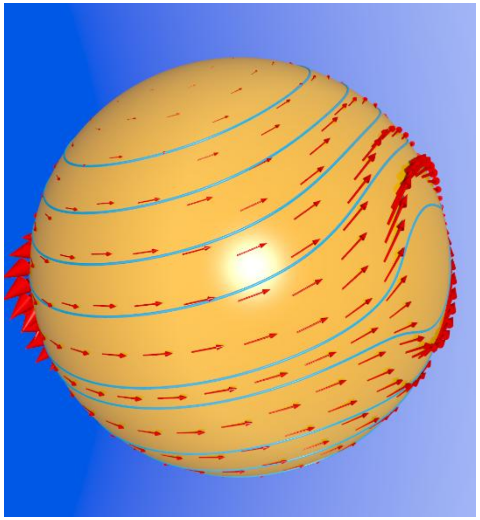
The tangent bundle of the sphere

Sphere S^2: Prominent example of a differential manifold

- Sphere plus tangent planes: tangent bundle TS^2
- TS^2 is example for a fiber bundle:

  Basis manifold M is S^2, fibers are the tangent planes T_pS^2
- Vector field: section of the fiber bundle





Parallel transport of vectors on S^2

- Comparison of different tangent spaces by parallel transport along a path
- Levi Civita connection
- Covariant derivative:

$$D_\lambda w := e_\mu\left(\partial_\lambda w^\mu + \Gamma^\mu_{\nu\lambda} w^\nu\right)$$

$$\Gamma^\mu_{\nu\lambda} : \text{Connection coefficients}$$

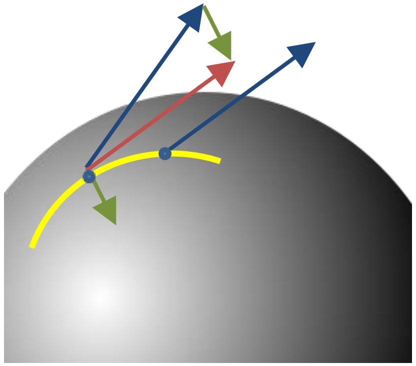
$$\text{Parallel transport}: \quad D_\lambda w = 0$$

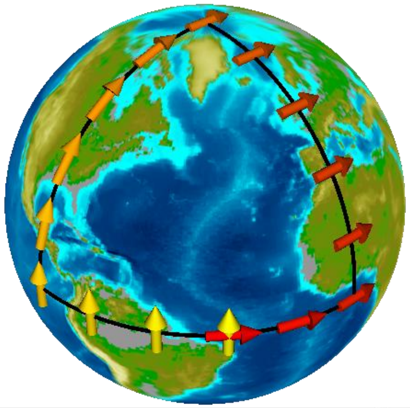
- Curvature: transport along closed path rotates vector (holonomy)
- Rotation angle:

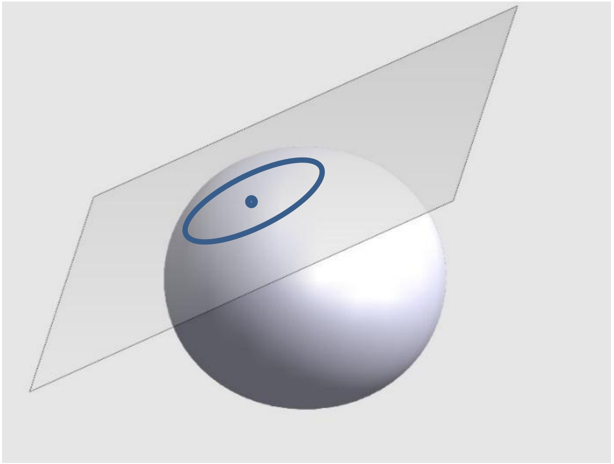
$$\Delta\omega = \int_{\substack{\text{encircled}\\ \text{area}}} d\Omega\, \kappa(\theta,\varphi)$$

$$\kappa = \text{Gaussian curvature}$$

- For description of parallel transport and curvature: attach rotation group SO(2) at each point, yields principal bundle $(S^2, SO(2)) = (S^2, S^1)$







Fiber bundles

- Consist of a basic differential manifold M
- At each point attached: fiber F_p which is either copy of a vector space ("vector bundle") or of a (gauge) group ("principle bundle")
- Prescriptions for glueing the fibers together e.g. Moebius strip (S^1,R)
- Prescription for parallel transport : covariant derivative and curvature

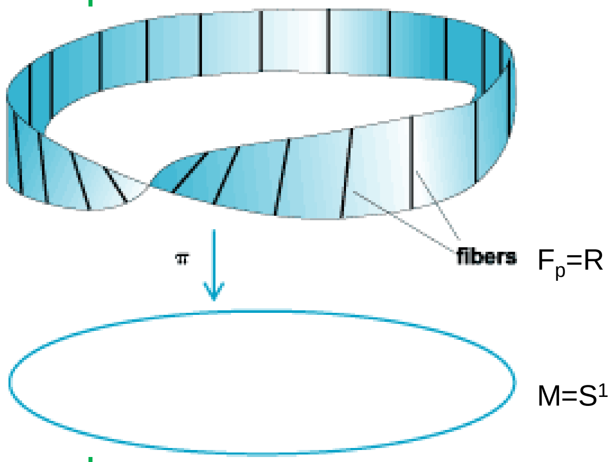
$$D_\lambda w := e_j\left(\partial_\lambda w^j + \Gamma^j_{i\lambda} w^i\right) = \left(\partial_\lambda + \underline{\underline{\Gamma}}_\lambda\right) w$$

$\underline{\underline{\Gamma}}_\lambda$: Connection coefficient matrix

Curvature tensor :

$$R^i_{j\mu\nu} = \partial_\mu \Gamma^i_{j\nu} - \partial_\nu \Gamma^i_{j\mu} + \Gamma^i_{l\mu}\Gamma^l_{j\nu} - \Gamma^i_{l\nu}\Gamma^l_{j\mu}$$

$$\underline{\underline{R}}_{\mu\nu} = \partial_\mu \underline{\underline{\Gamma}}_\nu - \partial_\nu \underline{\underline{\Gamma}}_\mu + \left[\underline{\underline{\Gamma}}_\mu, \underline{\underline{\Gamma}}_\nu\right]$$



- Topological quantum numbers, e.g. for 2d tangent bundles: Euler characteristic:

$$\chi = \frac{1}{2\pi} \int_{M^2} d\Omega\, \kappa(\theta,\varphi) = 2 - g$$

Fiber bundle in cosmology

$$T(R \oplus R^3) \text{ or } (R \oplus R^3, SO(1,3))$$

Fiber bundle in electrodynamics

Classical relativistic physics:

- Four momentum: $p^\mu = \begin{bmatrix} E/c \\ \vec{p} \end{bmatrix}$, four velocity : $v^\mu = p^\mu / m$
- In electromagnetic field : $v^\mu = \frac{1}{m}\left(p^\mu - qA^\mu\right)$, $A^\mu = \begin{bmatrix} \Phi \\ \vec{A} \end{bmatrix}$

Quantum mechanics:

- $\psi \in C$, i.e. section through a $\left(R \oplus R^3, C\right)$-bundle
- Parallelism?

$$p_\mu \to \frac{\hbar}{i}\partial_\mu,\quad v_\mu = \frac{\hbar}{im}\left(\partial_\mu - i\frac{q}{\hbar}A_\mu\right) \equiv \frac{\hbar}{im}D_\mu$$

$$D_\mu\psi = 0 \Rightarrow \psi = \psi_0 \exp\underbrace{\left\{i\frac{q}{\hbar}\int dx^\nu A_\nu\right\}}_{\text{phase change} \in U(1)}$$

- $\left(R \oplus R^3, U(1)\right)$ principle bundle

Topology:

- Curvature equals the electromagnetic field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \text{ cp. } \underline{\underline{R}}_{\mu\nu} = \partial_\mu \underline{\underline{\Gamma}}_\nu - \partial_\nu \underline{\underline{\Gamma}}_\mu + \left[\underline{\underline{\Gamma}}_\mu, \underline{\underline{\Gamma}}_\nu\right]$$

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$

- Topological quantum number: First Chern number

$$ch^{(1)} = \frac{1}{2\pi} \underbrace{\int dx^1 dx^2 F_{12}}_{\text{2d closed submanifold}}$$

Magnetic monopole of strength γ

- Vector potential and electromagnetic field tensor (acting on $\mathbb{C}$):

$$-iA_\theta = 0,\quad -iA_\varphi = \gamma i\cos\theta,\quad iF_{\theta\varphi} = \gamma i\sin\theta$$

- Cp. Levi-Civita connection on the sphere in orthonormal basis (acting on $\mathbb{R}^2$):

$$\underline{\underline{\hat{\Gamma}}}_\theta = 0,\quad \underline{\underline{\hat{\Gamma}}}_\varphi = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{i}\cos\theta,\quad \underline{\underline{\hat{R}}}_{\theta\varphi} = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{i}\sin\theta$$

- First chern number: $ch^{(1)} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, F_{\theta\varphi} = 2\gamma$

(R)

- $(\mathbb{R}\oplus\mathbb{R}^3, U(1))$: Electromagnetism
- $(\mathbb{R}\oplus\mathbb{R}^3, U(1)\otimes SU(2))$: Electroweak interaction
- $(\mathbb{R}\oplus\mathbb{R}^3, SU(3))$: Strong interaction

The Berry phase

- Quantum mechanical ground states dependent on parameter $\xi : |m(\xi)\rangle$
- $\xi \in M$ parameter manifold
- Example: Spin - 1/2 – particle in magnetic field of fixed strength, but arbitrary orientation : $M = S^2, \; \xi = (\vartheta, \varphi)$
- $|m(\xi)\rangle$ is determined up to a phase factor $e^{i\alpha}$, system is fiber bundle $(M, U(1))$

- Adiabatic motion and energy gap : particles remain in instantaneous state $|m(\xi)\rangle$
- Circular motion: phase change, holonomy
- Covariant derivative $D_\mu = \partial_\mu - iA_\mu$ with

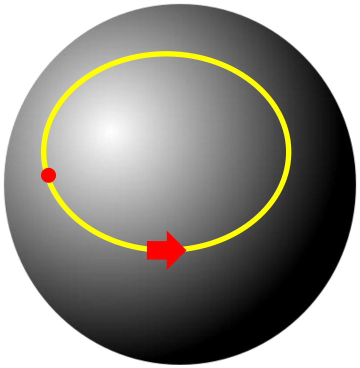
  Berry connection $A_\mu = i\langle m|\partial_\mu m\rangle$

  Berry curvature $F_{\mu\nu} = i\left\{\langle \partial_\mu m|\partial_\nu m\rangle - \langle \partial_\nu m|\partial_\mu m\rangle\right\}$
- First chern number for 2d parameter space

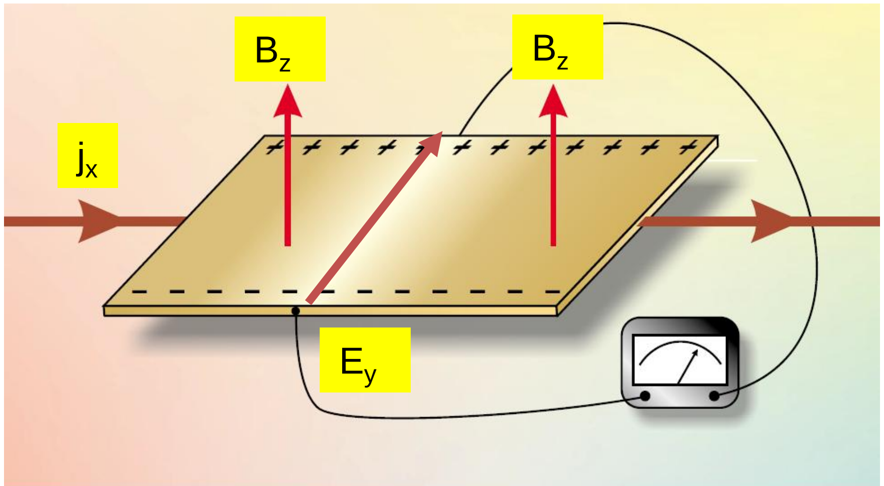
$$ch^{(1)}(M) = \frac{1}{2\pi}\int_M dx^1 dx^2 F_{12}$$

- Example spin - 1/2 - particle :

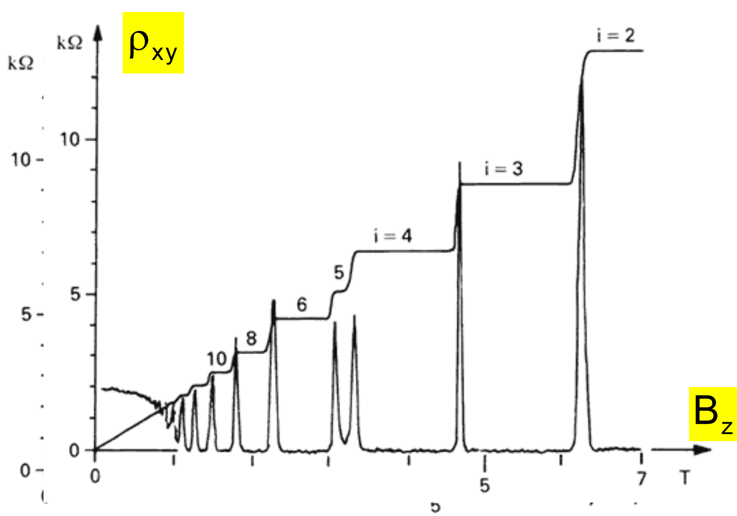
$$-iA_\theta = 0, \; -iA_\varphi = i\frac{1}{2}\cos\theta, \; iF_{\theta\varphi} = i\frac{1}{2}\sin\theta, \; ch^{(1)} = 1$$

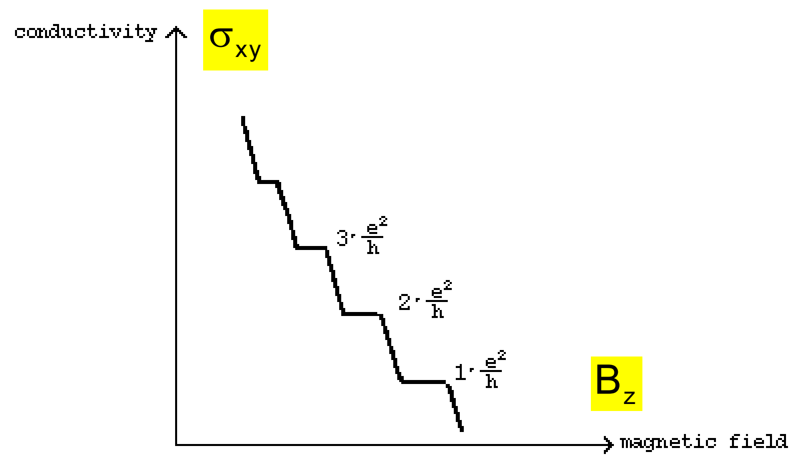


The Quantum Hall Effect (von Klitzing 1980)



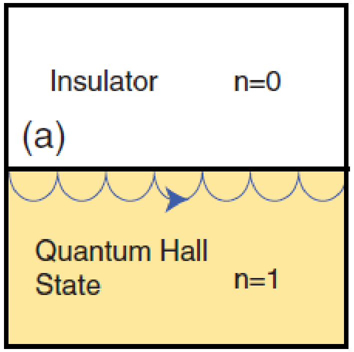
- 2d crystal in magnetic field
- $E_y = \rho_{yx} j_x \qquad j_x = \sigma_{xy} E_y$
- $\rho_{yx} = \frac{1}{n}\frac{h}{e^2} \qquad \sigma_{xy} = \frac{e^2}{h} n$



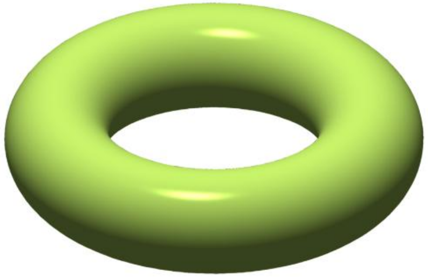


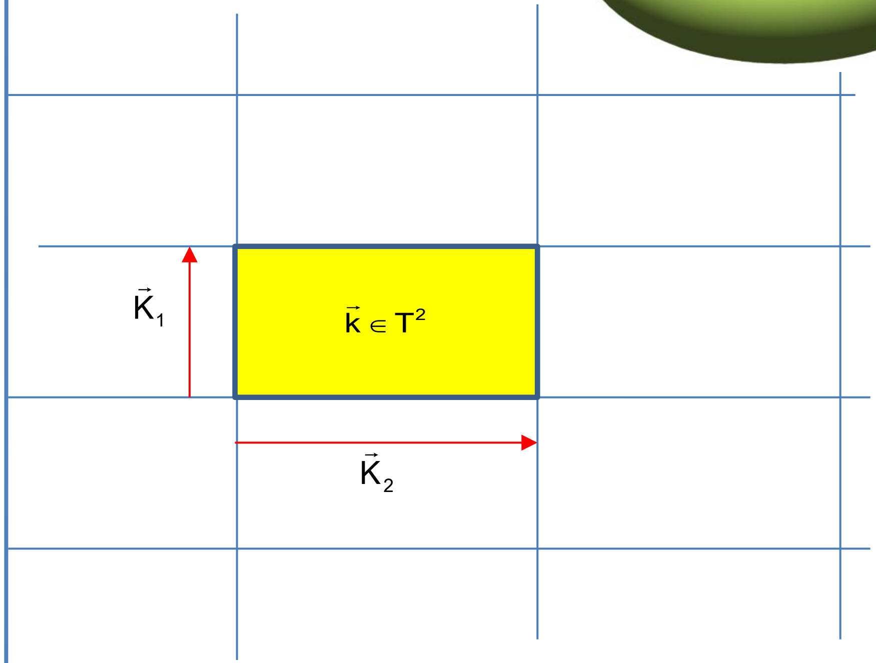
The Quantum Hall Effect

- Wave functions labelled by wavevector $\vec{k} = (k_1, k_2)$: $|u(\vec{k})\rangle$
- Wavevector space is also periodic: $\vec{k} \in R^2 \bmod (\vec{K}_1, \vec{K}_2) = T^2$
- Principle bundle $(T^2, U(1))$



Hasan MZ, Kane CL 2010 RMP 82, 3045





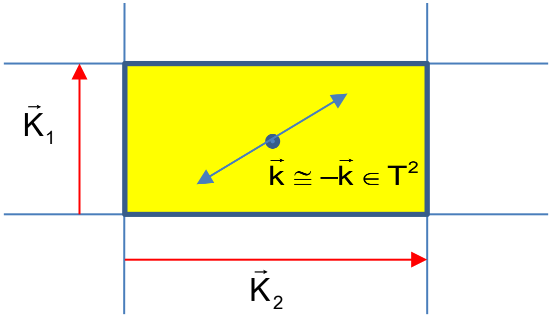
- Kubo transport formula for Hall conductivity :

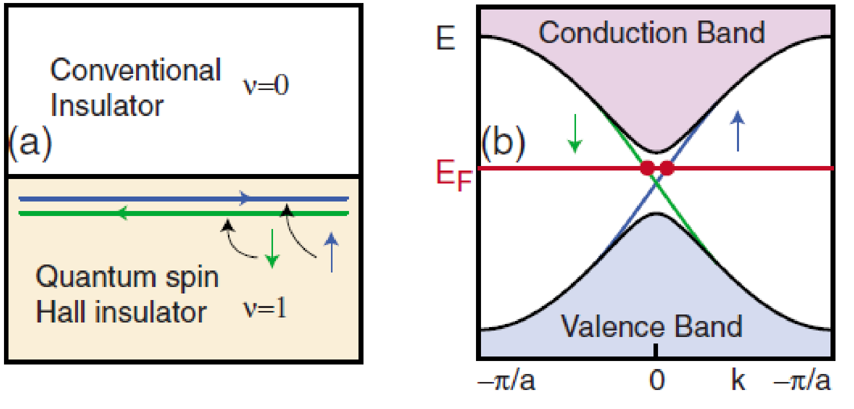
$$\sigma_{xy} = \frac{e^2}{h} \sum_{\substack{\text{occupied}\\ \text{bands}}} \frac{1}{2\pi} \int_{\text{Torus}} d^2k \; \underbrace{i\{\langle \partial_1 u | \partial_2 u \rangle - \langle \partial_2 u | \partial_1 u \rangle\}}_{\text{Berry curvature } F_{12}}$$

$$\sigma_{xy} = \frac{e^2}{h} \underbrace{\sum_{\substack{\text{occupied}\\ \text{bands}}} ch^{(1)}(T^2, U(1))}_{n}$$

The spin quantum Hall effect

- Restriction on the base manifold by time reversal symmetry: $\vec{k} \cong -\vec{k}$
- New base manifold $T^2 / \cong$
- New classification by $\nu \in Z_2 = \{0,1\}$
- HgTe/CdTe quantum well structures in 2d
- Spin polarized edge currents
- $Bi_{1-x}Sb_x$ in 3d, surface currents





Hasan MZ, Kane CL 2010 RMP 82, 3045

Summary

- Up to 1980: Quantum numbers based on symmetry
- Easy to break, lift of degeneracies
- Since 1980: Topological quantum numbers
- Robust

Congratulations and best wishes, Erik

Literature

- Hasan MZ, Kane CL 2010 Rev.Mod.Phys. 82, 3045
- Qi X-L, Chang S-Ch 2011 Rev.Mod.Phys. 83, 1057
- Budich JC, Trauzettel B 2013 phys.stat.sol.(RRL) 7, 109
- v. Klitzing K, Dorda G, Pepper M 1980 PRL 45, 494
- Thouless DJ, Kohmoto M, Nightingale MP, den Nijs M 1982 PRL 49, 405
- Kane CL, Mele EJ 2005 PRL 95, 146802
- Bernevig BA, Zhang S-Ch 2006 PRL 96, 106802
- Bernevig BA, Hughes DL, Zhang S-Ch 2006 Science 314, 1757
- König M, Wiedmann S, Brüne Ch, Roth A, Buhmann H, Molenkamp W, Qi X-L, Zhang S-Ch 2007 Science 318, 766
- Hsieh D, Qian D, Wray L, Xia Y, Hor YS, Cava RJ, Hasan MZ 2008 Nature 452, 970