The gravitational field of the rotating Earth versus the Kerr field in Einstein's theory of gravity

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Colloquium "Geodesy-Mathematics-Physics-Geophysics, Grafarend 75"

Herrn Kollegen H. Kautzleben sei herzlich für die Einladung gedankt.

See S. Kopeikin, M. Efroimsky, G. Kaplan, *Relativistic Celestial Mechanics of the Solar System,* Wiley-VCH (2011); C. Heinicke, FWH, *Schwarzschild and Kerr solutions of Einstein's field equation: An introduction,* Int. J. Mod. Phys. D 24, 1530006 (2015) *file ErikBerlin02.tex*

The gravitational field of the rotating Earth versus the Kerr field in Einstein's theory of gravity

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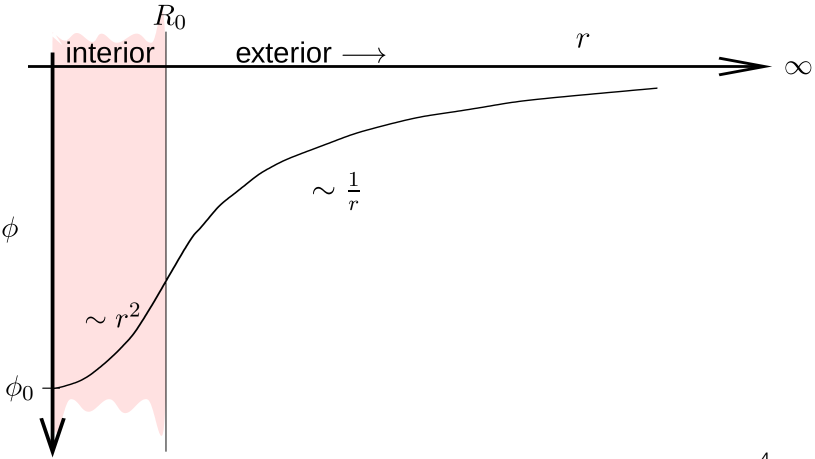
Erik Grafarend 75

- Ca. 48 years ago, Erik wrote his diploma thesis in physics at the Institute for Theoretical Physics of the Bergakademie (later Technical University) Clausthal. Earlier, he had studied mine surveying (Markscheidekunde).
- I was an assistant at that institute, helped Prof. Ekkehart Kröner to oversee Erik's diploma thesis. I had many discussions with Erik, even disputes... Erik was even sometimes right, as I now see with hindsight...
- Erik's thesis was related to nonlocal continuum mechanics (as partly developed by Kröner). The idea was to put this theory into a differential geometric framework, namely the so-called Kawaguchi geometry.
- The strain tensor ε_{ij} of a deformed continuum is prop. to the difference of the metric of the deformed and the undeformed continuum. Roughly we can say, for nonlocality the Lagrangian depends not only on strain ε but also on its derivatives $\mathcal{L} = \mathcal{L}(\varepsilon, \partial\varepsilon, \partial\partial\varepsilon, ...)$. The analogous is then true for the metric and the connection of the corr. Kawaguchi geometry:
- E. Grafarend, *Nonlocal continuum mechanics and Kawaguchi-geometry,* RAAG Research Notes, 3rd Ser., No. 141, 1–14 (August 1969) [Research Association of Applied Geometry (RAAG) Tokyo]; see T. Ootsuka..., *...Kawaguchi Lagrangian formulation,* arXiv:1406.2147.
- This seems to be a pattern in Erik's career: To take advantage of the most adv. formalisms of differential geometry and to apply it to geodesy.

Gravitation and electrodynamics

- **Newtonian gravity:**
  homogeneous (spherical) ball $\Phi \sim \frac{M}{r}$, *gravito*electric field
  the same ball rigidly rotating: the same potential! $\Phi \sim \frac{M}{r}$,
  no additional gravitomagnetic field.
  Newtonian potential of a homogeneous star:



- **Electromagnetism:**
  homogeneously electric charged (spherical) ball $\Phi_{\text{el}} \sim \frac{Q}{r}$
  electric field (Coulomb)
  the same ball rigidly rotating: Φ_{el} plus $\vec{A}_{\text{mg}} \sim \int \frac{dV \rho \vec{v}}{r}$
  magnetic field

- **Einsteinian gravity** (see Rindler, Relativity, Oxford 2001; $c = 1, G = 1$):
  homogeneous (spherical) massive rotating ball *in linear approximation:*

$$\Phi_{\text{GR}} = -\frac{m}{r}\,, \quad \vec{A}_{\text{GR}} = -\frac{2I\vec{\omega}\times\vec{r}}{r^3} \qquad (J = I\omega =: ma)$$

  J = modulus of angular momentum, I = moment of inertia, a angular momentum parameter.
  *gravito*electric $-\nabla\Phi_{\text{GR}}$, *gravito*magnetic field $\nabla\times\vec{A}_{\text{GR}}$, coo. (t, r, θ, ϕ):

$$ds^2 = \underbrace{-\left(1-\frac{2m}{r}\right)dt^2 + \frac{1}{1-\frac{2m}{r}}\,dr^2 + r^2\,\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)}_{\text{Schwarzschild 1916}} \underbrace{-\frac{4ma}{r}\,\sin^2\theta\, dt\, d\phi}_{\text{Lense-Thirring 1918}}$$

- The decisive difference to Newtonian gravity is the gravitomagnetic Lense-Thirring effect (LTE), which is proportional to the angular momentum (exper. verified by LAGEOS satellites and Gravity Probe B).
  M. Zimbres, Class. Quantum Grav. 31, 215 006 (2014). "...the Earth model plays a significant role in the multipole contribution to the LTE."

Exact Kerr vacuum solution

Not quite seriously: “Schwarzschild” (left) versus “Kerr” (right)

.....in Boyer-Lindquist (Schwarzschild like) coordinates (t, r, θ, ϕ):

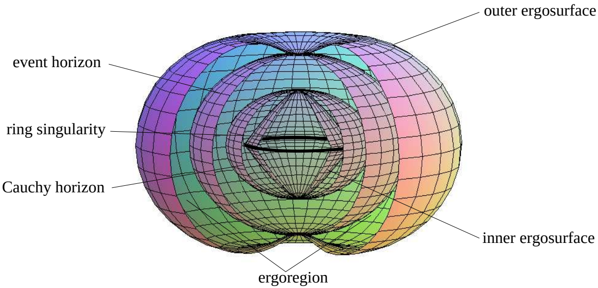
$$ds^2 = -\left(1 - \frac{2mr}{\rho^2}\right) dt^2 - \frac{4mar\sin^2\theta}{\rho^2}\, dt\, d\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2ma^2 r\sin^2\theta}{\rho^2}\right)\sin^2\theta\, d\phi^2$$

$\rho^2 := r^2 + a^2\cos^2\theta\,, \quad \Delta := r^2 - 2mr + a^2 = (r - r_+)(r - r_-)\,, \quad r_\pm := m \pm \sqrt{m^2 - a^2}$

Mass $M = m/G$, angular momentum J, with $a = J/M$ as ang. m. parameter

Ergosurfaces, horizons, and singularities for *slow* and *fast* Kerr black holes

a<m



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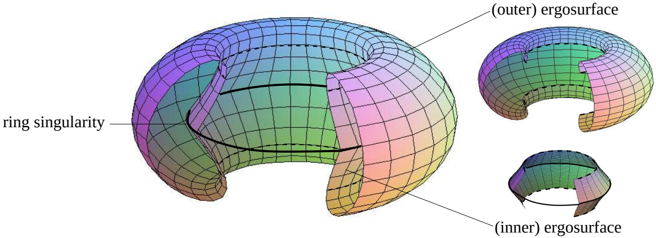

Kerr solution and gravitomagnetism, multipole moments

- Kerr metric with the 20 independent curvature components (“tidal forces”) can be represented by a trace-free symmetric 6×6 curvature matrix R_{AB}. The collective indices $A, B, .. = 1, ..., 6$ are defined as follows: $\{\hat{t}\hat{r}, \hat{t}\hat{\theta}, \hat{t}\hat{\phi}; \hat{\theta}\hat{\phi}, \hat{\phi}\hat{r}, \hat{r}\hat{\theta}\} \longrightarrow \{1, 2, 3; 4, 5, 6\}$. Then,

curv. matrix: $$(R_{AB}) = \begin{pmatrix} -2\mathbb{E} & 0 & 0 & 2\mathbb{B} & 0 & 0 \\ 0 & \mathbb{E} & 0 & 0 & -\mathbb{B} & 0 \\ 0 & 0 & \mathbb{E} & 0 & 0 & -\mathbb{B} \\ 2\mathbb{B} & 0 & 0 & 2\mathbb{E} & 0 & 0 \\ 0 & -\mathbb{B} & 0 & 0 & -\mathbb{E} & 0 \\ 0 & 0 & -\mathbb{B} & 0 & 0 & -\mathbb{E} \end{pmatrix} = (R_{BA})$$

- Here

$$\mathbb{E} := mr \frac{r^2 - 3a^2 \cos^2\theta}{(r^2 + a^2\cos^2\theta)^3}\,, \qquad \mathbb{B} := ma\cos\theta \frac{3r^2 - a^2\cos^2\theta}{(r^2 + a^2\cos^2\theta)^3}\,.$$

It is straightforward to identify $\mathbb{E}$ as the gravitoelectric and $\mathbb{B}$ as the gravitomagnetic tidal field component of the curvature.

- This is an exact result of (nonlinear) GR, see Jens Boos, arXiv.org: 1412.1958 (2014).

- Geroch & Hansen(1970/74) defined within GR multipole moments for static and stationary systems. Thorne (1980) defined moments via a *post-Newtonian approach* for slowly changing (the case of the Earth!) as well as for stationary systems.

  Y. Gürsel, *...The equivalence of the Geroch-Hansen formulation and the Thorne formulation*, General Relativity and Gravitation 15, 737 (1983).

  Geodesists and satellite-flying people measure the *zonal and tesseral spherical harmonics* of the Earth's gravitational field. They are equivalent to the Geroch-Hansen and Thorne multipole moments.

- The Geroch-Hansen multipole moments for the Kerr solution:

$s=0$	$M_0 = -m$	(mass)	$J_1 = \ \ ma$ (angular momentum)
$s=1$	$M_2 = \ \ ma^2$	(mass quadrupole)	$J_3 = -ma^3$
$s=2$	$M_4 = -ma^4$		$J_5 = \ \ ma^5$
$s=3\dots$	$M_6 = \ \ ma^6\dots$		$J_7 = -ma^7\dots$

  Thus, we have for the gravitoelectric and the gravitomagnetic moments

$$M_{2s} = (-1)^{s+1} m\, a^{2s}\,, \qquad J_{2s} = 0\,,$$

$$M_{2s+1} = 0\,; \qquad J_{2s+1} = (-1)^{s} m\, a^{2s+1}\,.$$

- Apparently, the Kerr metric has a simple *multipolar* structure, depending only on m and a, or, formulated differently, only very specific matter distributions can represent the interior of the Kerr metric.

Gravitational field of the rotating Earth

- $R_\oplus = (\mathtt{a}^2\mathtt{c})^{\frac{1}{3}} = 6.37... \times 10^6$ m (mean radius, equatorial r. $\mathtt{a}$, polar r. $\mathtt{c}$)
- $M_\oplus = 5.97... \times 10^{24}$ kg (mass)
  $GM_\oplus = 3.98... \times 10^{14}$ m^3 s^{-2} (...multiplied by gravitational constant)
- $J_\oplus = \omega C = 5.85... \times 10^{33}$ m^2 kg s^{-1} ('spin' angular momentum)
- $m_\oplus := \frac{GM_\oplus}{c^2} \approx 4$ mm, $r_{\oplus\mathrm{Sch}} \approx 8$ mm, $a_\oplus := \frac{J_\oplus}{M_\oplus c} \approx 3$ m, thus
  $\frac{a_\oplus}{m_\oplus} \approx 700$, corr. to over-extreme Kerr solution with naked singularity
  (black hole Sgr A*, in contrast, has (?) $\frac{a}{m} \approx 0.7 \rightarrow$ event horizon)
- $\eta = \frac{GM_\oplus}{c^2 R_\oplus} \approx 0.7 \times 10^{-9}$ and $\varepsilon = \left(\frac{\omega_\oplus R_\oplus}{c}\right)^2 \approx 2.3 \times 10^{-12}$ are the two parameters for relativistic corrections
- Simplified expression for the Earth's gravitational potential

$$\Phi \approx \frac{GM_\oplus}{r}\left[1 - \sum_{l=2}^{\infty}\left(\frac{\mathtt{a}}{r}\right)^l J_l P_l sin\phi\right] , \qquad P_l = \text{Legendre polynomials}$$

Zonal coeffs. J_l in units 10^{-6}, with $J_2 = 1082.6, \;\; J_3 = -2.5, \;\; J_4 - 1.6...$

- A. N. Cox (ed.) *Allen's Astrophyical Quantities,* 4th ed., Springer 2000
- S. M. Kopeikin et al., *Exact relativistic theory of geoid's undulation,* Phys. Lett. A 2014/15 (in press)

The q-metric as a prototyp for the Earth?

- For axial symmetry, no Birkhoff theorem exists. A multitude of axially symmetric solutions is known to exist, see, e.g., H. Quevedo and B. Mashhoon, *Generalization of Kerr spacetime,* PRD 43, 3902 (1991).
- As an example, S. Toktarbay and H. Quevedo, *A stationary q-metric,* Grav. & Cosmolgy 20, 252-254 (2014) describe a solution with mass m, quadrupole moment q and angular momentum a. For the Geroch-Hansen moments, they find (for some parameter σ)

$$
\begin{aligned}
M_0 &= m + \sigma q\,, \quad M_2 = \frac{7}{3}\sigma^3 q - \frac{1}{3}\sigma^3 q^3 + m\sigma^2 - m\sigma^2 q^2 - 3m^2\sigma q - m^3, \\
J_1 &= ma + 2a\sigma q\,, \\
J_3 &= -\frac{1}{3}a(-8\sigma^3 q + 2\sigma^3 q^3 - 3m\sigma^2 + 9m\sigma^2 q^2 + 12m^2\sigma q + 3m^3).
\end{aligned}
$$

  Even gravitomagnetic and odd gravitoelectric multipoles vanish. Higher odd gravitomagnetic and even gravitoelectric multipoles are linearly dependent: they are determined by the parameters m, a, σ, and q.
- The stationary q-metric can be used to describe the exterior gravitational field of a rotating deformed mass distribution. We have to imagine that the gravitational field of the Earth could be matched to some kind of similar solution of Einstein's equation. Highly nontrivial!

Special and general relativity in geodesy

- **GPS**: N. Ashby, *Relativity and the global positioning system,* Physics Today 55, 41–47 (May 2002):

  "...satellite speeds v are about 4 km/s. Time dilation then causes the moving clocks' frequencies to be slow by $\Delta f/f = v^2/2c^2 \approx 10^{-10}$. Gravitational effects are even larger. In fact, relativistic effects are about 10 000 times too large to ignore."

  "... An effect of comparable size is contributed by the gravitational blueshift, which results when a photon-or a clock-moves to lower altitude. If these relativistic effects were not corrected for, satellite clock errors building up in just one day would cause navigational errors of more than 11 km, quickly rendering the system useless." (emphases by the speaker)

- **Geodetic surveying**: Since GPS is an integral part of modern geodetic surveying, it is apparent that relativistic effects will play an increasing role therein. This will be demonstrated in the subsequent slide:

Future

Collaborative Research Center SFB 1128 (J. Flury, Hannover, et al.):
Relativistic Geodesy and Gravimetry with Quantum Sensors
— Modeling, Geo-Metrology, and Future Technology — (geo-Q)

- Example: Clock network modeling for relativistic geodesy by C. Lämmerzahl (Bremen), J. Müller (Hannover), D. Puetzfeld (Bremen) "... new generation of optical clocks connected by optical fiber links for a variety of applications such as Earth gravity field determination, time comparison or synchronization, navigation and positioning, and also fundamental physics tests."
- GRACE and follow-ups

Literature:

- S. Kopeikin, M. Efroimsky, G. Kaplan, *Relativistic Celestial Mechanics of the Solar System,* Wiley-VCH (2011)
- D. Puetzfeld, C. Lämmerzahl, B. F. Schutz, eds., *Equations of Motion in Relativistic Gravity,* Fundamental theories of Physics, Springer (...2015)
- E. Poisson, C. M. Will, *Gravity: Newtonian, Post-Newtonian, Relativistic,* Cambridge (2014)

♣ I am grateful to Sergei Kopeikin (Columbia, Missouri) for advice and to Dirk Puetzfeld (Bremen) for helpful remarks. And I wish Erik & Ulrike all the best for the future.

"The gravitational field of the rotating Earth versus the Kerr field in Einstein's theory of gravity"

by Friedrich W. Hehl, Univ. of Cologne and Univ. of Missouri, Columbia

Abstract

Already at the beginning of his career, Erik Grafarend was interested in non-Riemannian geometry, namely in Kawaguchi geometry, in differential geometry generally and, specifically, in the calculus of exterior differential forms. He always wanted to have the best mathematical tools for mine-surveying, for application in physics and in geodesy. If we consider the rotating Earth together with its gravitational field—and this is of central interest in any institute of geodesy—then the best available description is within the framework of general relativity. For most practical purposes, the gravitational field of the rotating Earth is needed only up 2nd post-Newtonian approximation. However, in this short eulogy on Erik, I will display the exact Kerr vacuum solution of general relativity, which depends on the 2 parameters mass and angular momentum. In particular, I will discuss the gravitomagnetic field of the Kerr solution, which vanishes in Newtonian approximation. Some dis/similarities between the Kerr solution and the rotating Earth will be touched.

Friday, 13 February 2015, 11:00–11:30h