Leibniz Society of Science at Berlin, Scientific Colloquium
Geodesy - Mathematic - Physics - Geophysics
in honour of Erik W. Grafarend
on the occasion of his 75th birthday
Berlin, Germany, 13 February 2015

Summation of Series and an Approximation of Legendre's Functions in Constructing Integral Kernels for the Exterior of an Ellipsoid: Application to Boundary Value Problems in Physical Geodesy

Petr Holota

Research Institute of Geodesy, Topography and Cartography
Zdiby, Prague-East, Czech Republic

1. Kernels in the Classical Solution

In geodesy we are dealing with a number of interesting problems of mathematical physics. We follow ***classical as well as weak solution*** concept and often use an ***integral representation of the solution***, ***Green's function method*** or also ***Galerkin's approximations*** of the solution.

The classical example is the famous ***Stokes function*** $G(\boldsymbol{x},\boldsymbol{y})$. It is associated with the problem to find T such that

$$\frac{\partial T}{\partial |\boldsymbol{x}|}+\frac{2}{R}T=f \quad \text{for} \quad |\boldsymbol{x}|=R \qquad \text{and} \qquad \Delta T=g \quad \text{for} \quad |\boldsymbol{x}|>R$$

Apart from the first degree harmonics $T_1(\boldsymbol{x})$ the function enables to get the solution of the problem in the following form

$$T(\boldsymbol{x})=T_1(\boldsymbol{x})-\frac{1}{4\pi}\int_{|\boldsymbol{y}|=R}G(\boldsymbol{x},\boldsymbol{y})\,f(\boldsymbol{y})\,d_yS-\frac{1}{4\pi}\int_{|\boldsymbol{y}|>R}G(\boldsymbol{x},\boldsymbol{y})g(\boldsymbol{y})d\boldsymbol{y}$$

For the exterior of the sphere of radius R ***Stokes's function*** can even be nicely expressed in a closed form. We have

$$G(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{R}{|\boldsymbol{x}|}\frac{1}{|\bar{\boldsymbol{x}}-\boldsymbol{y}|} - 3\frac{R|\bar{\boldsymbol{x}}-\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|^2}$$

$$-\frac{R^3\cos\psi}{|\boldsymbol{x}|^2|\boldsymbol{y}|^2}\left[5+3\ln\frac{1}{2}\left(1-\frac{R^2\cos\psi}{|\boldsymbol{x}||\boldsymbol{y}|}+\frac{|\bar{\boldsymbol{x}}-\boldsymbol{y}|}{|\boldsymbol{y}|}\right)\right]$$

where

$$\bar{\boldsymbol{x}} = \frac{R}{|\boldsymbol{x}|^2}\boldsymbol{x} \quad \textbf{and} \quad |\boldsymbol{x}||\bar{\boldsymbol{x}}| = R^2$$

and ψ is the angle between the position vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

***Remark 1*.**

$G(\boldsymbol{x},\boldsymbol{y})$ **above is a more general function in comparison with the function usually applied in physical geodesy.**

It enables to solve the geodetic boundary value problem not *only for Laplace's* equation but also for *Poissons' partial differential equation*.

Another example is ***Neumann's function*** $N(\boldsymbol{x}, \boldsymbol{y})$. Their use has now reasonable motivations in physical geodesy. It is associated with the problem to find T such that

$$\frac{\partial T}{\partial |\boldsymbol{x}|} = f \quad \text{for} \quad |\boldsymbol{x}| = R \qquad \text{and} \qquad \Delta T = g \quad \text{for} \quad |\boldsymbol{x}| > R$$

We also have a closed form expression for this function

$$N(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{R}{|\boldsymbol{x}|} \; \frac{1}{|\bar{\boldsymbol{x}}-\boldsymbol{y}|} - \frac{1}{R} \ln \frac{|\bar{\boldsymbol{x}}-\boldsymbol{y}| + |\bar{\boldsymbol{x}}| - |\boldsymbol{y}|\cos\psi}{|\boldsymbol{y}|(1-\cos\psi)}$$

Remark 2. For $|\boldsymbol{y}| = R$, it is obvious that $N = N(\boldsymbol{x}, \psi)$ and we easily deduce that

$$N = \frac{2}{|\boldsymbol{x}-\boldsymbol{y}|} - \frac{1}{R} \ln \frac{|\boldsymbol{x}-\boldsymbol{y}| + R - |\boldsymbol{x}|\cos\psi}{|\boldsymbol{x}|(1-\cos\psi)}$$

In the case that $|\boldsymbol{x}| = |\boldsymbol{y}| = R$, we even have

$$N = N(\psi) = \frac{1}{R}\left[\frac{1}{\sin(\psi/2)} - \ln\frac{1+\sin(\psi/2)}{\sin(\psi/2)}\right]$$

2. Reproducing Kernel and the Weak Solution

In geodesy kernel functions have their position also within the *concept of the weak solution* of boundary value problems.

Good example is the *reproducing kernel*. Nevertheless, its construction corresponds not only to the *geometry of the solution domain*, but also to the *scalar product associated* with the particular *Hilbert's space* of functions used.

Consider e.g. the space $H_2^{(1)}(S_R)$ of functions harmonic in the exterior of the sphere of radius R which is equipped with the following scalar product

$$(u,v)_1 = A(u,v)\,, \quad \text{where} \quad A(u,v) = \sum_{i=1}^{3} \int_{S_R} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, d\mathbf{x}$$

One can show that there exists a function $K(\mathbf{x},\mathbf{y})$ such that

$$\sum_{i=1}^{3}\int_{S_R}\frac{\partial K(\boldsymbol{x},\boldsymbol{y})}{\partial x_i}\frac{\partial v(\boldsymbol{x})}{\partial x_i}\,d\boldsymbol{x} = v(\boldsymbol{y})$$ **holds for all** $v \in H_2^{(1)}(S_R)$

and that in this case

$$K(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi R}\sum_{n=0}^{\infty}\frac{2n+1}{n+1} z^{n+1} P_n(\cos\psi) \quad , \qquad z = \frac{R^2}{|\,\boldsymbol{x}\,||\,\boldsymbol{y}\,|}$$

where P_n **is the usual *Legendre polynomial* of degree** n**, and** ψ **is the angle between the position vectors** $\boldsymbol{x}$ **and** $\boldsymbol{y}$ **.**

Moreover, it is not extremely difficult to find that

$$K(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi R}\left(\frac{2z}{L} - \ln\frac{L+z-\cos\psi}{1-\cos\psi}\right)$$

where

$$L = \sqrt{1-2z\cos\psi + z^2}$$

The advantage of using the kernel function $K(\boldsymbol{x}, \boldsymbol{y})$ can be seen from the construction of *Galerkin's approximations* to the solution of *Neumann's problem*.

Recall, therefore, that in this case we are looking for u such that

$$\Delta u = 0 \quad \text{in} \quad S_R \qquad \text{and} \qquad \frac{\partial u}{\partial n} = -f \quad \text{on} \quad \partial S_R$$

where Δ is *Laplace's operator* and $\partial / \partial n$ denotes the derivative in the direction of the unit (outer) normal $\boldsymbol{n}$ of ∂S_R.

***In the numerical solution* the function u is approximated by the linear combination**

$$u_n = \sum_{j=1}^{n} c_j^{(n)} v_j$$

where v_j are members of a function basis of $H_2^{(1)}(S_R)$.

The coefficients $c_j^{(n)}$ can be obtained from the respective *Galerkin system*

$$\sum_{j=1}^{n} c_j^{(n)} A(v_j, v_k) = \int_{\partial S_R} v_k \; f \; dS \;, \qquad k = 1, \ldots, n$$

where

$$A(v_j, v_k) = \sum_{i=1}^{3} \int_{S_R} \frac{\partial v_j}{\partial x_i} \frac{\partial v_k}{\partial x_i} \, d\boldsymbol{x}$$

Putting now

$$v_j(\boldsymbol{x}) = K(\boldsymbol{x}, \boldsymbol{y}_j)$$

we can immediately deduce that *in our Galerkin system the elements* $A(v_j, v_k)$ are given by

$$A(v_j, v_k) = K(\boldsymbol{y}_j, \boldsymbol{y}_k)$$

3. Use of Elementary Potentials

The advantage of using the function $K(\boldsymbol{x},\boldsymbol{y})$ definitely attracts attention. Nevertheless, it is also tempting to use *elementary potentials* for constructing *Galerkin's approximations*.

This means that the basis functions will be of the following form

$$v_j(\boldsymbol{x}) = \frac{1}{|\boldsymbol{x}-\boldsymbol{y}_j|}$$

In this case one can show that the *diagonal elements* are given by

$$A(v_j, v_j) = \pi\left(\frac{2R}{R^2 - |\boldsymbol{y}_j|^2} + \frac{1}{|\boldsymbol{y}_j|}\ln\frac{R+|\boldsymbol{y}_j|}{R-|\boldsymbol{y}_j|}\right)$$

Concerning the *off diagonal elements*, however, the *computation is more laborious*.

In particular, we have to use a series representation of $v_j(\boldsymbol{x})$ **, i.e.**

$$v_j(\boldsymbol{x}) = \sum_{n=0}^{\infty} \frac{|\boldsymbol{y}_j|^n}{|\boldsymbol{x}|^{n+1}} P_n(\cos\psi_j)$$

This enables us to get

$$A(v_j, v_k) = \frac{4\pi}{R} \sum_{n=0}^{\infty} \frac{n+1}{2n+1} z^n P_n(\cos\psi_{jk}) \quad \textbf{where} \quad z = \frac{|\boldsymbol{y}_j||\boldsymbol{y}_k|}{R^2}$$

or

$$A(v_j, v_k) = \frac{2\pi}{R} \sum_{n=0}^{\infty} z^n P_n(\cos\psi_{jk}) + \frac{4\pi}{R} \sum_{n=0}^{\infty} \frac{1}{2n+1} z^n P_n(\cos\psi_{jk})$$

Here, however, we are faced by the *problem to find a closed expression for the off diagonal element* $A(v_j, v_k)$

… and what we only can get is

$$A(v_j, v_k) = \frac{2\pi}{R\sqrt{1-2z\cos\psi_{jk} + z^2}} + \frac{4\pi}{R}S$$

where

$$S = \frac{1}{\sqrt{z}} \int \frac{1}{\sqrt{z\left(1-2z\cos\psi_{jk} + z^2\right)}} dz$$

is an *elliptic integral*.

4. Transition to an Ellipsoid of Revolution

We have seen that the mathematical apparatus related to the exterior of the sphere is fairly developed and transparent. However, the sphere is rather far from reality, i.e. from the real surface of the Earth. ***Therefore, our aim is discuss the construction of the apparatus for the exterior of an oblate ellipsoid of revolution.***

We will suppose that a **and** b**,** $a \geq b$**, are the semi-axes of an oblate ellipsoid of revolution and will consider the ellipsoidal coordinates** u, β, λ **related to** x_1**,** x_2**,** x_3 **by the equations**

$$x_1 = \sqrt{u^2 + E^2} \cos\beta \cos\lambda \,, \quad x_2 = \sqrt{u^2 + E^2} \cos\beta \sin\lambda \,, \quad x_3 = u \sin\beta$$

where $E = \sqrt{a^2 - b^2}$ **denotes the** ***linear eccentricity*****, see e.g.** ***Heiskanen and Moritz (1967)*****. Clearly, the boundary** $\partial\Omega_{ell}$ **is then defined by** $u = b$**.**

Going now back to $K(\boldsymbol{x},\boldsymbol{y})$, or more precisely to $K_{ell}(\boldsymbol{x},\boldsymbol{y})$, and referring to *Holota (2004, 2011,2014)*, we can deduce that

$$K_{ell}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi b}\sum_{n=0}^{\infty}(2n+1)\Big[K_{n0xy}P_n(\sin\beta_x)P_n(\sin\beta_y) + $$

$$+ 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}K_{nmxy}\,P_{nm}(\sin\beta_x)P_{nm}(\sin\beta_y)\cos m(\lambda_x-\lambda_y)\Big]$$

with

$$K_{nmxy} = \frac{iEb}{a^2}\,\frac{Q_{nm}(z_x)}{Q_{nm}(z_0)}\,\frac{Q_{nm}(z_y)}{Q_{nm}(z_0)}\left[\frac{1}{Q_{nm}(z_0)}\,\frac{dQ_{nm}(z_0)}{dz}\right]^{-1}$$

where P_{nm} and Q_{nm} are associated *Legendre's functions* of the 1st and the 2nd kind, respectively; while

$$z_x = \frac{iu_x}{E}\;, \quad z_y = \frac{iu_y}{E}\;, \quad z_0 = \frac{ib}{E} \quad \text{and} \quad i = \sqrt{-1}$$

The problem, however, is the summation of the series **representing the kernel** $K_{ell}(\boldsymbol{x}, \boldsymbol{y})$**.**

In this connection, inspecting the right hand side of

$$K_{nmxy} = \frac{iEb}{a^2} \frac{Q_{nm}(z_x)}{Q_{nm}(z_0)} \frac{Q_{nm}(z_y)}{Q_{nm}(z_0)} \left[\frac{1}{Q_{nm}(z_0)} \frac{dQ_{nm}(z_0)}{dz} \right]^{-1}$$

it is tempting to use the *limit layer theory* as discussed in *Sona (1995)* or *Sansò and Sona (2001)* and also analyzed in *Holota (2003a)*.

5. Legendre's Differential Equation

As it is known

$$v_{nm}(u) = \frac{Q_{nm}(z)}{Q_{nm}(z_0)} = Q_{nm}\left(i\frac{u}{E}\right) \Big/ Q_{nm}\left(i\frac{b}{E}\right)$$

is a solution of

$$(u^2 + E^2)\frac{d^2 v_{nm}}{du^2} + 2u\frac{dv_{nm}}{du} - \left[n(n+1) - \frac{E^2 m^2}{u^2 + E^2}\right] v_{nm} = 0$$

which can be shown to be equivalent to *Legendre's equation* by using the pure imaginary variable $z = iu/E$**.**

This equation has two solutions:

***Legendre's functions* of the 1st kind** P_{nm} **and**
***Legendre's functions* of the 2nd kind** Q_{nm}

The functions Q_{nm} **are the suitable solutions in case that we deal with harmonic functions in an *unbounded domain*.**

6. Approximation – Limit Layer Theory

As in *Sona (1995)* we will work now with the variable $s = u/b$.

It is obvious that $v_{nm}(u) = v_{nm}(sb) = v(s)$ and we can deduce that for v *Legendre's equation transforms* into

$$(s^2 + e^2)\frac{d^2v}{ds^2} + 2s\frac{dv}{ds} - \left[n(n+1) - \frac{e^2m^2}{s^2+e^2}\right]v = 0 \text{ , where } e^2 = \frac{E^2}{a^2}$$

Now, confining ourselves to $s \in \langle 1, s_{\max} \rangle$, where $s_{\max}$ is an upper bound for s, such that

$$e^2 s^{-2} \approx e^2 \quad \text{and} \quad (s^2 + e^2)^{-1} \approx (1+e^2)^{-1}$$

can be taken for admissible approximations, we can simplify the equation above. It transforms into

$$(1+e^2)s^2\frac{d^2w}{ds^2} + 2s\frac{dw}{ds} - \left[n(n+1) - \frac{e^2m^2}{1+e^2}\right]w = 0$$

The solution $w(s)$ obviously differs from $v(s)$. Following *Sona (1995)*, we have

$$w(s)=\frac{A}{s^{\alpha}}$$ **where A *is a constant***

and α has to be *one of the two roots*, $\alpha_1 \approx -n$ and $\alpha_2 \approx n+1$, of the quadratic equation

$$(1+e^2)\alpha^2-(1-e^2)\alpha-n(n+1)+\frac{e^2m^2}{1+e^2}=0$$

Clearly, α_2 is the suitable solution in our case and we obtain

$$w(s)=\frac{1}{s^{n+1-\varepsilon}}\quad , \quad \varepsilon=e^2\frac{(n+1)(n+2)+m^2}{2n+1}$$

where A has been fixed so that $w(1)=1$.

***Note in particular* that $v(s)=w(s)=1$ for $s=1$.**

As regards the differenced of the two solutions, it has been estimated by means of *Gronwall's inequality* in *Holota (2003a)*.

7. Coefficients in the Expansion of $K_{ell}(x,y)$

Within the limit layer theory, **we can write approximately that**

$$\frac{Q_{nm}(z_x)}{Q_{nm}(z_0)} \approx \left(\frac{b}{u_x}\right)^{n+1-\varepsilon} \quad , \quad \frac{Q_{nm}(z_y)}{Q_{nm}(z_0)} \approx \left(\frac{b}{u_y}\right)^{n+1-\varepsilon}$$

and subsequently derive that

$$\frac{1}{Q_{nm}(z_0)} \frac{dQ_{nm}(z_0)}{dz} \approx \frac{E}{ib} \frac{dw(1)}{ds} = \frac{iE}{b}(n+1-\varepsilon)$$

which holds, but again with some accuracy only.

Now using some algebra, the right hand side of the equation above can be substantially modified. We obtain

$$\frac{1}{Q_{nm}(z_0)} \frac{dQ_{nm}(z_0)}{dz} \approx i\frac{Eb}{a^2}(n+1)\left[1+\frac{E^2}{ab}\frac{(n+1)(n-1)-m^2}{(n+1)(2n+1)}\right]$$

Thus in

$$K_{ell}(\mathbf{x},\mathbf{y}) = \frac{1}{4\pi b}\sum_{n=0}^{\infty}(2n+1)\Big[K_{n0xy}P_n(\sin\beta_x)P_n(\sin\beta_y) + $$

$$+ 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}K_{nmxy}\,P_{nm}(\sin\beta_x)P_{nm}(\sin\beta_y)\cos m(\lambda_x-\lambda_y)\Big]$$

we will have

$$K_{nmxy} \approx \left(\frac{b^2}{u_x u_y}\right)^{n+1-\varepsilon}\kappa_{nm} \approx \left(\frac{b^2}{u_x u_y}\right)^{n+1}\left[1-\varepsilon\ln\frac{b^2}{u_x u_y}\right]\kappa_{nm}$$

with

$$\varepsilon = \left(\frac{E}{a}\right)^2\frac{(n+1)(n+2)+m^2}{2n+1}$$

and

$$\kappa_{nm} = \frac{1}{n+1}\left[1-\frac{E^2}{ab}\,\frac{(n+1)(n-1)-m^2}{(n+1)(2n+1)}\right]$$

Now, denoting by ψ the angular distance of points (β_x, λ_y) and (β_y, λ_y) on a sphere, when β and λ are interpreted as spherical latitude and longitude, respectively and using the well-known *Legendre's addition theorem*, we can deduce that

$$K_{ell}(\boldsymbol{x},\boldsymbol{y}) \approx \frac{1}{4\pi b} K^{(1)}(\boldsymbol{x},\boldsymbol{y}) - \frac{E^2}{4\pi ab^2} K^{(2)}(\boldsymbol{x},\boldsymbol{y}) + \frac{E^2}{4\pi ab^2} K^{(3)}(\boldsymbol{x},\boldsymbol{y}) -$$

$$- \frac{E^2}{4\pi a^2 b}\left(\ln \frac{b^2}{u_x u_y}\right) K^{(4)}(\boldsymbol{x},\boldsymbol{y}) + \frac{E^2}{4\pi a^2 b}\left(\ln \frac{b^2}{u_x u_y}\right) K^{(5)}(\boldsymbol{x},\boldsymbol{y})$$

with

$$K^{(1)}(\boldsymbol{x},\boldsymbol{y}) = \sum_{n=0}^{\infty} \frac{2n+1}{n+1}\left(\frac{b^2}{u_x u_y}\right)^{n+1} P_n(\cos\psi)$$

$$K^{(2)}(\boldsymbol{x},\boldsymbol{y}) = \sum_{n=0}^{\infty} \frac{n-1}{n+1} \left(\frac{b^2}{u_x u_y} \right)^{n+1} P_n(\cos\psi)$$

$$K^{(3)}(\boldsymbol{x},\boldsymbol{y}) = -\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \left(\frac{b^2}{u_x u_y} \right)^{n+1} \frac{\partial^2 P_n(\cos\psi)}{\partial^2 \lambda}$$

$$K^{(4)}(\boldsymbol{x},\boldsymbol{y}) = \sum_{n=0}^{\infty} (n+2) \left(\frac{b^2}{u_x u_y} \right)^{n+1} P_n(\cos\psi)$$

and

$$K^{(5)}(\boldsymbol{x},\boldsymbol{y}) = \sum_{n=1}^{\infty} \frac{1}{n+1} \left(\frac{b^2}{u_x u_y} \right)^{n+1} \frac{\partial^2 P_n(\cos\psi)}{\partial^2 \lambda}$$

For terms containing the derivative of P_n **with respect to** λ ***Holota (2003b)*** **has been used.**

8. An Alternative Concept for our Approach

In our approach we so far used an *exact solution of an approximate differential equation*. Let us try now to change this philosophy and work with an *approximate solution of an exact differential equation*.

In this case we recall that

$$Q_{nm}(z) =$$

$$= (-1)^m \frac{2^n n!(n+m)!}{(2n+1)!}(z^2-1)^{-\frac{n+1}{2}} F\left(\frac{n+m+1}{2}, \frac{n-m+1}{2}, \frac{2n+3}{2}; \frac{1}{1-z^2}\right)$$

where F **is a *hypergeometric function* and that**

$$F_z = F\left(a, b, c; \frac{1}{1-z^2}\right) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n\, n!} \left(\frac{1}{1-z^2}\right)^n$$

with $(a)_n = a(a+1)\ldots(a+n-1),\ n = 1,2,3\ldots$ **[similarly for** $(b)_n$ **and** $(c)_n$**].**

After elementary modifications we have

$$F_z = F_{z_0} + \frac{ab}{c}\left(\frac{1}{1-z^2} - \frac{1}{1-z_0^2}\right) +$$

$$+ \frac{a(a+1)b(b+1)}{2c(c+1)}\left[\left(\frac{1}{1-z^2}\right)^2 - \left(\frac{1}{1-z_0^2}\right)^2\right] + \cdots$$

Thus, e.g., in a $20km$ **layer close above the ellipsoid**

$$\frac{1}{1-z^2} - \frac{1}{1-z_0^2} \leq e^4 = 0,000046$$

and

$$\left(\frac{1}{1-z^2}\right)^2 - \left(\frac{1}{1-z_0^2}\right)^2 \leq 2e^6 = 0,0000006$$

etc. In the sequel, therefore, considering these estimates, we can simply put

$$\frac{Q_{nm}(z_x)}{Q_{nm}(z_0)}\frac{Q_{nm}(z_y)}{Q_{nm}(z_0)} \approx \rho^{n+1}$$ **where** $$\rho = \frac{a^2}{\sqrt{u_x^2+E^2}\sqrt{u_y^2+E^2}}$$

Similarly recalling that

$$\frac{dQ_{nm}}{dz} = \frac{(n+1)\,z}{1-z^2}Q_{nm} - \frac{n-m+1}{1-z^2}Q_{n+1,m}$$

and putting

$$\frac{Q_{n+1,m}(z_0)}{Q_{nm}(z_0)} \approx \frac{n+m+1}{2n+3}\frac{E}{ia}$$

we arrive at

$$\frac{1}{Q_{nm}(z_0)}\frac{dQ_{nm}(z_0)}{dz} \approx i\frac{Eb}{a^2}(n+1)\left[1+\frac{E^2}{ab}\frac{(n+1)^2-m^2}{(n+1)(2n+3)}\right]$$

Hence

$$K_{nmxy} \approx \rho^{n+1}\kappa_{nm}$$ **with** $$\kappa_{nm} = \frac{1}{n+1}\left[1-\frac{E^2}{ab}\frac{(n+1)^2-m^2}{(n+1)(2n+3)}\right]$$

Now we return to the reproducing kernel $K_{ell}(\boldsymbol{x},\boldsymbol{y})$.
Denoting again by ψ the angular distance of points (β_x,λ_y) and (β_y,λ_y) on a sphere, when β and λ are interpreted as spherical latitude and longitude, respectively, we can deduce that

$$K_{ell}(\boldsymbol{x},\boldsymbol{y}) \approx \frac{1}{4\pi b}K^{(1)}(\boldsymbol{x},\boldsymbol{y}) - \frac{E^2}{4\pi ab^2}K^{(2)}(\boldsymbol{x},\boldsymbol{y}) + \frac{E^2}{4\pi ab^2}K^{(3)}(\boldsymbol{x},\boldsymbol{y})$$

with

$$K^{(1)}(\boldsymbol{x},\boldsymbol{y}) = \sum_{n=0}^{\infty}\frac{2n+1}{n+1}\rho^{n+1}P_n(\cos\psi)$$

$$K^{(2)}(\boldsymbol{x},\boldsymbol{y}) = \sum_{n=0}^{\infty}\frac{2n+1}{2n+3}\rho^{n+1}P_n(\cos\psi)$$

$$K^{(3)}(\boldsymbol{x},\boldsymbol{y}) = -\sum_{n=1}^{\infty}\frac{2n+1}{(n+1)^2(2n+3)}\rho^{n+1}\frac{\partial^2 P_n(\cos\psi)}{\partial^2\lambda}$$

see *Holota (2011, 2004)*. For the expression of the last term *Holota (2003b)* has been used.

Thank you
for your attention !