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Boundary problems of mathematical physics in Earth's gravity field studies

Abstract

The theory of boundary value problems for Laplace's and Poisson's equation offers a natural basis for gravity field studies, especially in case they rest on terrestrial measurements. Free, fixed and mixed boundary value problems are considered. Concerning the linear problems, the classical as well as the weak solution concept is applied. Some techniques are shown. Also an attempt is made to construct the respective Green's function, reproducing kernel and entries in Galerkin's matrix for the solution domain given by the exterior of an oblate ellipsoid of revolution. The integral kernel is expressed by series of ellipsoidal harmonics and its summation is discussed. Some aspects associated with the formulation of boundary value problems in gravity field studies based on terrestrial gravity measurements in combination with satellite data on gravitational field are mentioned too.

1. Introduction

Studies on Earth's gravity field enable to learn more about our planer. The motivation considered here comes primarily from geodetic applications. We particularly focus on the related mathematics and mathematical tools. Potential theory has a special position in Earth's gravity field studies, but other braches of mathematics are of great importance too. The theory of boundary value problems for elliptic partial differential equations of second order, in particular Laplace's and Poisson's equation, offer a natural basis for gravity field studies, especially in case they rest on terrestrial measurements.

In the determination of the gravity potential W and figure of the Earth from terrestrial gravity, levelling and astrogeodetic data one has to solve a free boundary-value problem for Laplace's (or Poisson's) equation. This concept in principle covers already the results by George Gabriel Stokes published in 1849 in his work "On the variation of gravity on the surface of the Earth". Nevertheless, only in the second half of the last century the free boundary nature of the problem resulted in an explicit mathematical formulation. This development is represented especially by the work of M.S. Molodensky and his group, see (Molodensky, Eremeev and Yurkina, 1962). Also the problem itself is usually called Molodensky's problem. Nevertheless, other names are firmly connected with this concept too, in particular Pellinen, Brovar, Moritz, Krarup, Bursa, Pick and others. More details can be found, e.g, in (Heiskanen and Moritz, 1967), (Moritz, 1977), (Moritz, 1980) and also in (Holota, 1977). In 1976 the process culminated in the famous paper (Hörmander, 1976), where the solution in terms of functional analysis is presented with an exceptional rigor. In this work the nonlinearity of the problem was treated explicitly and the tie between geodesy and mathematics was demonstrated very clearly. Comments on this are also in (Holota, 1980). Hörmander refers especially to (Krarup, 1973), but also to (Moritz, 1972). Though an idealized situation is assumed in his work, the physical model is explicitly defined. The Earth is considered a rigid body. The data measured on the surface of the Earth are corrected for gravitational interaction with the Moon, the Soon and planets, for precession and nutation of the Earth and so on. The theoretical development in mid-seventies was rather rich and it is also associated with the work by Sansò. In his papers the concept of gravity space was successfully applied for the solution of Molodensk's problem, see (Sansò, 1977, 1978a, 1978b).

Nevertheless, in practice the approach to the geodetic free boundary value problem mostly confines to the solution of its linearized version. Our explanation starts with the use of the method of integral equations and with the Green's function method. Subsequently, in addition to the classical solution the weak solution concept and variational methods are considered. The approach is more flexible and also some historical facts concerning the weak solution concept are mention.

In the following discussion the impact of space geodetic measurements is taken into consideration. It changes the free boundary nature of the problem into the fixed gravimetric boundary value problem. Nevertheless, for the nature of the boundary condition the problem is nonlinear too, though again it is treated most frequently in its linear version only. More details can be found in (Holota, 1997). The weak solution is represented by a linear combination of suitable basis functions. This leads to Galerkin approximations and the solution of large systems of linear equations. Special attention is paid to basis functions generated by the reproducing kernel in the respective spaces of harmonic functions.

The complex structure of the Earth's surface makes the solution of the boundary problems considered rather demanding. This equally concerns the classical as well as the weak solution. Some techniques, e.g. the transformation of the solution domain and successive approximations, that may solve these difficulties, are shown. Also an attempt is made to construct the reproducing kernels for solution domain given by the exterior of an oblate ellipsoid of revolution. The kernel is expressed by series of ellipsoidal harmonics and its summation is discussed.

The final notes are devoted to some considerations associated with the role of boundary value problems in gravity field studies based on terrestrial gravity measurements combined with space geodetic and space gravitational field data. Mixed boundary value problems are mentioned and also an optimization approach is considered since in the majority of cases the problems to be solved are overdetermined by nature.

2. Linear problem

In solving the linear version of the geodetic free boundary value problem we start with an assumption that a potential U represents a model of the real gravity potential W of the Earth and that a surface Σ (telluroid) is in one-to-one correspondence with the surface of the Earth and approximates its figure. We usually work in a system of rectangular Cartesian coordinates x_1, x_2, x_3 such that its origin is in the center of gravity of the Earth and its x_3 axes coincides with the rotation axes of the Earth. The problem is then to find the disturbing potential $T = W - U$ such that

$$\Delta T = 0 \quad \text{outside } \Sigma \tag{1}$$

and

$$T + \langle \boldsymbol{h}, \boldsymbol{grad}\, T \rangle = \Delta W + \langle \boldsymbol{h}, \Delta \boldsymbol{g} \rangle \quad \text{on } \Sigma, \tag{2}$$

where Δ means Laplace's differential operator, $\langle .,. \rangle$ is the scalar product, $\boldsymbol{h} = -\boldsymbol{M}^{-1}\boldsymbol{\gamma}$, $\boldsymbol{\gamma} = \boldsymbol{grad}\, U$ and $\boldsymbol{M}_{ij} = \partial^2 U / \partial x_i \partial x_j$ for $i, j = 1,2,3$. The input data are represented by the gravity anomaly Δg and the potential anomaly ΔW. (Here definitely Δ will not be confused with Laplace's operator). The anomaly Δg is the difference between $\boldsymbol{g} = \boldsymbol{grad}\, W$ measured on the surface of the Earth and γ in the corresponding point on the surface Σ. Similarly ΔW represents the difference between W resulting from leveling combined with gravimetric measurements on the surface of the Earth and U in the corresponding point on the surface Σ.

Inspecting the problem quickly we immediately see that it is an oblique derivative problem. In addition, for physical reasons, we have to assume that the solution T meats the following asymptotic condition at infinity

$$T = \frac{c}{|\boldsymbol{x}|} + O\left(|\boldsymbol{x}|^{-3}\right) \quad \text{for } |\boldsymbol{x}| \to \infty, \tag{3}$$

where $|\boldsymbol{x}| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ and O denotes the Landau symbol. The position anomaly $\boldsymbol{\zeta}$ (improvement of the surface Σ) then results from

$$\boldsymbol{\zeta} = \boldsymbol{M}^{-1}\left(\Delta \boldsymbol{g} - \boldsymbol{grad}\, T\right). \tag{4}$$

Note. In is worth mentioning that in practice many important results in the solution of the boundary problem above where obtained for its special interpretation in terms of the so-called spherical approximation (considered as a mapping), see (Moritz, 1980). In this case the problem is to find T such that

$$\Delta T = 0 \quad \text{outside } \Gamma \tag{5}$$

and

$$|\boldsymbol{x}| \frac{\partial T}{\partial |\boldsymbol{x}|} + 2T = 2\Delta W - |\boldsymbol{x}|\, \Delta g \quad \text{on } \Gamma, \tag{6}$$

where Γ represents the image of Σ under the spherical approximation (mentioned above) and Δg is the magnitude of $\Delta \boldsymbol{g}$. At infinity again an asymptotic condition as in Eq. (3) is prescribed for the solution T.

3. Problems associated with the use of integral equation method

The method of integral equations is a classical technique for the solution of boundary value problems of potential theory. Starting with Molodensky its application for the solution of the linear geodetic boundary value problem is widely discussed in physical geodesy and in the literature dealing with mathematical problems in gravity field studies, see e.g. (Heiskanen and Moritz, 1967). If applied for the solution of the problem given by Eqs. (5) and (6), one represents T by a single-layer potential

$$T(\boldsymbol{x}) = \int_{\Gamma} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \mu(\boldsymbol{y})\, d_y \Gamma \tag{7}$$

and looks for the unknown density μ. This leads to the following integral equation

$$\begin{aligned} 2\pi \mu(\boldsymbol{x}) \cos(\boldsymbol{x}, \boldsymbol{n}_x) &- \frac{3}{2|\boldsymbol{x}|} \int_{\Gamma} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \mu(\boldsymbol{y})\, d_y \Gamma - \\ &- \frac{1}{2|\boldsymbol{x}|} \int_{\Gamma} \frac{|\boldsymbol{y}|^2 - |\boldsymbol{x}|^2}{|\boldsymbol{y}-\boldsymbol{x}|^3} \mu(\boldsymbol{y})\, d_y \Gamma = \Delta g \,, \end{aligned} \tag{8}$$

where $\boldsymbol{n}$ denotes the outer unit normal of the surface Γ. Starting with Molodensky a method of "shrinking parameter" is usually applied for constructing a series representation of its solution, see also (Moritz, 1973). However, some specific problems are associated with the solution of this integral equation. This in particular concerns the relation to Riesz' theorem on compact operator, and the application of the well-know Fredholm alternatives. The cause is the fact that second integral has a strongly singular kernel and thus does not exist in the usual Lebesgue sense, see (Mikhlin, 1962).

4. Cauchy's Principal Value – Example

An integral defined in Lebesgue's sense does not depend on how the partition of the integration domain is made finer. It represents a common value of the infimum of the upper and of the supremum of the lower Lebesgue sums. The idea cannot be applied to integrals with strongly singular kernels. In our case, therefore, the strongly singular integral can only be computed as Cauchy's principal value, usually symbolized by v.p. ("valeur principale")

$$\text{v.p.} \int_{\Gamma} \frac{|\boldsymbol{y}|^2 - |\boldsymbol{x}|^2}{|\boldsymbol{y}-\boldsymbol{x}|^3} \mu(\boldsymbol{y}) \, d_y \Gamma = \lim_{\varepsilon \to 0} \int_{\Gamma - \Gamma_{x,\varepsilon}} \frac{|\boldsymbol{y}|^2 - |\boldsymbol{x}|^2}{|\boldsymbol{y}-\boldsymbol{x}|^3} \mu(\boldsymbol{y}) \, d_y \Gamma \,, \tag{9}$$

where $\Gamma_{\boldsymbol{x},\varepsilon} \equiv \{\boldsymbol{y} \in \Gamma; |\boldsymbol{y} - \boldsymbol{x}| \leq \varepsilon\}$.

However, we are faced by some peculiar behavior of an integral defined in the sense like this. Together with the fact that it does not exist in the usual Lebesgue sense, its special properties can be also illustrated by the following example in one-dimensional analogy. Let's assume that $x \in (a,b)$ and define e.g. a function

$$\begin{aligned} f(x) &= \text{v.p.} \int_a^b \frac{d\xi}{\xi - x} = \lim_{\varepsilon \to 0} \int_a^{x-\varepsilon} \frac{d\xi}{\xi - x} + \lim_{\varepsilon \to 0} \int_{x+\varepsilon}^b \frac{d\xi}{\xi - x} = \\ &= \lim_{\varepsilon \to 0} \ln \frac{\varepsilon}{x-a} + \lim_{\varepsilon \to 0} \ln \frac{b-x}{\varepsilon} = \ln \frac{b-x}{x-a} \end{aligned} \tag{10}$$

It is obvious that

$$f'(x) = \frac{a-b}{(x-a)(b-x)} \quad \text{and} \quad f''(x) = \frac{(b-a)(a+b-2x)}{(a-x)^2 (b-x)^2}. \tag{11}$$

However, changing the order of integration and derivation, we do not get the same result for the first derivative, while the result for the second derivative coincides. Indeed,

$$\text{v.p.} \int_a^b \frac{d}{dx} \left(\frac{1}{\xi - x} \right) d\xi = - \text{ p.v.} \int_a^b \frac{d\xi}{(\xi - x)^2} = - \infty, \tag{12}$$

$$\text{v.p.} \int_a^b \frac{d^2}{dx^2} \left(\frac{1}{\xi - x} \right) d\xi = \text{p.v.} \int_a^b \frac{2 \, d\xi}{(\xi - x)^3} = \frac{(b-a)(a+b-2x)}{(a-x)^2 (b-x)^2}. \tag{13}$$

5. Green's function method

Green's function is an important tool in solving boundary value problems associated with ordinary as well as partial differential equations. In our applications we will denote this function by $G(\boldsymbol{x}, \boldsymbol{y})$. It gives us a possibility to explicitly represent the solution not only for Laplace's, but also for Poisson's equation. Green's function can easily be constructed in cases that the solution domain has an elementary shape. Indeed, suppose that we have a constant $R > 0$ and consider e.g. the following problem

$$\Delta T = g \quad \text{for } |\boldsymbol{x}| > R \tag{14}$$

and

$$\frac{\partial T}{\partial |\boldsymbol{x}|} + \frac{2}{R} T = f \quad \text{for } |\boldsymbol{x}| = R. \tag{15}$$

Following the standard principles in constructing Green's functions, we obtain

$$\begin{aligned} G(\boldsymbol{x}, \boldsymbol{y}) \;=\; & \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} + \frac{R}{|\boldsymbol{x}|} \frac{1}{|\bar{\boldsymbol{x}} - \boldsymbol{y}|} - \frac{3R\,|\bar{\boldsymbol{x}} - \boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|^2} - \\ & - \frac{R^3 \cos\psi}{|\boldsymbol{x}|^2 |\boldsymbol{y}|^2} \left[5 + 3 \ln \frac{1}{2} \left(1 - \frac{R^2 \cos\psi}{|\boldsymbol{x}||\boldsymbol{y}|} + \frac{|\bar{\boldsymbol{x}} - \boldsymbol{y}|}{|\boldsymbol{y}|} \right) \right], \end{aligned} \tag{16}$$

where $\bar{\boldsymbol{x}} = \left(R^2 / |\boldsymbol{x}| \right) \boldsymbol{x}$ is given by an inversion in a sphere, see e.g. (Holota, 1985, 1995, 2003). As mentioned, the function $G(\boldsymbol{x}, \boldsymbol{y})$ makes it possible to express the solution of our boundary value problem explicitly. The natural point of departure is a (slightly modified) Green's third identity

$$\begin{aligned} T(\boldsymbol{x}) \;=\; & -\frac{1}{4\pi} \int_{|\boldsymbol{y}| > R} G(\boldsymbol{x}, \boldsymbol{y}) \Delta T(y)\, d\boldsymbol{y} \;- \\ & - \frac{1}{4\pi} \int_{|\boldsymbol{y}| = R} \left[G(\boldsymbol{x}, \boldsymbol{y}) \frac{\partial T(\boldsymbol{y})}{\partial |\boldsymbol{y}|} - T(\boldsymbol{y}) \frac{\partial G(\boldsymbol{x}, \boldsymbol{y})}{\partial |\boldsymbol{y}|} \right] d_y S\,, \end{aligned} \tag{17}$$

where $d\boldsymbol{y} = dy_1 dy_2 dy_3$ means the volume element. For $G(\boldsymbol{x}, \boldsymbol{y})$ as above the formula immediately yields

$$T(\boldsymbol{x}) = T_1(\boldsymbol{x}) - \frac{1}{4\pi} \int_{|\boldsymbol{y}| = R} G(\boldsymbol{x}, \boldsymbol{y}) f(\boldsymbol{y})\, d_y S - \frac{1}{4\pi} \int_{|\boldsymbol{y}| > R} G(\boldsymbol{x}, \boldsymbol{y}) g(\boldsymbol{y})\, d\boldsymbol{y}, \tag{18}$$

so that $T(\boldsymbol{x})$ is determined uniquely apart from first degree harmonic components $T_1(\boldsymbol{x})$. Nevertheless, recall that $T_1(\boldsymbol{x})$ are eliminated through the asymptotic condition given by Eq. (3).

Note. The restriction of $G(\boldsymbol{x}, \boldsymbol{y})$ for $|\,\boldsymbol{y}\,| = R$ or $|\,\boldsymbol{x}\,| = |\,\boldsymbol{y}\,| = R$ attains the form of the famous (Pizzetti extended) Stokes function or the classical Stokes function, respectively. Both these restrictions of the original function $G(\boldsymbol{x}, \boldsymbol{y})$ are well-known in physical geodesy, see (Heiskanen and Moritz, 1967).

6. Green's function and a more general boundary

Green's function $G(\boldsymbol{x}, \boldsymbol{y})$ mentioned above is essentially associated with the fact that the problem given by Eqs. (14) and (15) is considered for a sphere of radius R. Nevertheless, $G(\boldsymbol{x}, \boldsymbol{y})$ may be applied also in a more general case. This can be shown by means of a transformation of coordinates,

$$y_i = y_i(x_1, x_2, x_3), \, i = 1, 2, 3, \tag{19}$$

that gives the geodetic boundary-value problem represented by Eqs. (5) and (6) the structure of the simple problem treated in Section 5, i.e.

$$\varDelta_y T = g \quad \text{for } |\,\boldsymbol{y}\,| > R \tag{20}$$

and

$$\frac{\partial T}{\partial |\,\boldsymbol{y}\,|} + \frac{2}{R} T = f^* \quad \text{for } |\,\boldsymbol{y}\,| = R \tag{21}$$

with the only substantial difference that

$$g = g(T) = \delta^{ij} \frac{\partial^2 T}{\partial y_i \partial y_j} - g^{ij} \left(\frac{\partial^2 T}{\partial y_i \partial y_j} - \varGamma_{ij}^k \frac{\partial T}{\partial y_k} \right), \tag{22}$$

where $\delta^{ij} = 0$, for $i \neq j$, while the metric tensor g^{ij} and the Christoffel symbols $\varGamma_{ij}^k$ depend on the geometry of the original boundary T. Thus in the curvilinear coordinates y_1, y_2, y_3 the second term on the right hand side of Eq. (22) represents the Laplacian applied on T, while the first term has the structure of the Laplacian that corresponds to y_1, y_2, y_3 formally interpreted as orthogonal coordinates.

Clearly, the representation formula (18) now changes into an integro-differential equation

$$T(\boldsymbol{y}) = T_1(\boldsymbol{y}) - \frac{1}{4\pi}\int_{|\xi|=R} G(\boldsymbol{y},\xi)\, f^*(\xi)\, d_\xi S - \\ - \frac{1}{4\pi}\int_{|\xi|>R} G(\boldsymbol{y},\xi)\, g\left[T(\xi), \frac{\partial T(\xi)}{\partial \xi_i}, \frac{\partial^2 T(\xi)}{\partial \xi_i \partial \xi_j}\right] d\xi\,. \qquad (23)$$

In both the cases, i.e. when the integral equation method or Green's function method is used, the solution leads to successive approximations. Nevertheless, for the method that rests on Greens' function all the integrals involved exist in the usual Lebesgue sense. More details on the convergence of the approximations applied for the solution of Eq. (23) may be found in (Holota, 1985, 1989, 1992a, 1992b).

7. Weak solution

As demonstrated in the preceding sections the approach to boundary-value problems in gravity field studies often represents what is known as the classical solution. We look for a smooth function satisfying the differential equation and the boundary condition "pointwise". The method of integral equations and Green's function method are most frequently used. Alternatively, we can look for a measurable function satisfying a certain integral identity connected with the boundary-value problem in question. This is the so-called weak solution. Natural function spaces corresponding to this method are Sobolev's spaces.

In many cases there even exists a possibility to replace the integration of a differential equation under given boundary conditions by an equivalent problem of getting a function that minimizes some integral. This corresponds to variational methods.

Historically, the first use of variational methods was in the form of Dirichlet's principle. According to this principle: among functions which attain given values on the boundary $\partial\Omega$ of a domain Ω, that and only that function which is harmonic in Ω minimizes the so-called Dirichlet's integral

$$\int_{\Omega} \sum_{i=1}^{3} \left(\frac{\partial u}{\partial x_i}\right)^2 d\boldsymbol{x}\,, \qquad (24)$$

see (Michlin, 1970), (Rektorys, 1974), (Kellogg, 1953) and others. Dirichlet's principle was extensively used by Riemann (1826-1866), but critically commented by Weierstrass (1815-1897) and later by Hadamard (1865-1963). In the beginning of the 20th century the principle got a new interest. Hilbert (1862-1943) showed that the justification of Dirichlet's principle is essentially associated with the notion of the completeness of the metric space. The justification by Hilbert has a close tie to his contributions to the development of the calculus of variations, see Giaquinta (2000).

Return now to geodesy and recall that our intention is to discuss problems associated with the determination of the external gravity field of the Earth. In consequence we have to suppose that Ω mentioned above is an unbounded solution domain. Therefore, as regards functions spaces, we will work with functions from Sobolev's weighted space $W_2^{(1)}(\Omega)$ endowed with inner product

$$(u,v)_1 \equiv \int_\Omega \frac{uv}{|\boldsymbol{x}|^2}\, d\boldsymbol{x} + \sum_{i=1}^{3} \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, d\boldsymbol{x}, \tag{25}$$

see (Holota, 1997). Also the boundary $\partial\Omega$ of the domain Ω will be supposed to have a certain degree of regularity. Putting $\Omega' = \boldsymbol{R}^3 - \Omega \cup \partial\Omega$, we will suppose that Ω' is a domain with the so-called Lipschitz (or Lipschitz regular) boundary, see (Necas, 1967), (Rektory, 1974) or (Kufner et al., 1977). Lipschitz regularity represents a considerably weaker assumption in comparison with the classical formulation, where the boundary is usually assumed to be at least two times continuously differentiable.

8. Example – Neumann's problem and a quadratic functional

Put

$$A(u,v) = \sum_{i=1}^{3} \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, d\boldsymbol{x}, \tag{26}$$

which is a bilinear form on $W_2^{(1)}(\Omega) \times W_2^{(1)}(\Omega)$ and consider the quadratic functional

$$\Phi(u) = A(u,u) - 2 \int_{\partial\Omega} uf\, dS \tag{27}$$

defined on $W_2^{(1)}(\Omega)$, where f belongs to the space $L_2(\partial\Omega)$ of square integrable functions on $\partial\Omega$. The functional Φ attains its minimum in $W_2^{(1)}(\Omega)$, which results from the theory of an abstract variational problem, see e.g. (Necas and Hlavacek, 1981) and (Holota, 2000, 2004). Conversely, assuming that Φ has its local minimum at a point $u \in W_2^{(1)}(\Omega)$, we necessarily arrive at

$$A(u,v) = \sum_{i=1}^{3} \int_{\Omega} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, d\boldsymbol{x} = \int_{\partial\Omega} vf\, dS \tag{28}$$

valid for all $v \in W_2^{(1)}(\Omega)$. This integral identity represents Euler's necessary condition for Φ to have a minimum at the point u. It has also a classical interpretation. Under some regularity assumptions one can apply Green's identity and show that u has to be a solution of Neumann's (exterior) problem, i.e.,

$$\Delta u = 0 \quad \text{in} \quad \Omega \tag{29}$$

and

$$\frac{\partial u}{\partial n} = -f \quad \text{on} \quad \partial\Omega, \tag{30}$$

where $\partial/\partial n$ denotes the derivative in the direction of the unit normal $\boldsymbol{n}$ of $\partial\Omega$.

9. An oblique derivative problem

As already indicated, in Earth's gravity field studies we are faced by a rather complex physical reality. In this connection the use of the weak solution concept has some advantages. It is more flexible. We could demonstrate it for the linear version of the geodetic free boundary value problem mentioned in Section 2. A more amplified discussion can be also found in (Holota, 1999) or (Holota and Nesvadba 2007a).

Nevertheless, considering recent developments in geodesy, we will prefer another example. Its importance is justified by the fact that nowadays there is no need to solve a free boundary value problem to obtain the disturbing potential T, as the geometry of the boundary may be obtained from space geodetic measurements. As a result, confining our discussion still to linear setting, we can focus on the solution of the following gravimetric fixed boundary value problem with an oblique derivative. This means we will lock for T such that

$$\Delta T = 0 \quad \text{in} \quad \Omega \tag{31}$$

and

$$\left\langle \boldsymbol{s}, \boldsymbol{grad}\, T \right\rangle = -\,\delta g \quad \text{on} \quad \partial\Omega, \tag{32}$$

where the solution domain Ω is the exterior of the Earth, $\partial\Omega$ represents its boundary and $\boldsymbol{s} = -\left(1/\gamma\right)\boldsymbol{grad}\, U$ with γ denoting the magnitude of $\boldsymbol{\gamma} = \boldsymbol{grad}\, U$. Moreover, as a harmonic function T is assumed regular at infinity, i.e. $T = O\left(|\boldsymbol{x}|^{-1}\right)$ for $\boldsymbol{x} \to \infty$. The input date are represented by the gravity disturbance δg which, in contrast to the free boundary value problem, is a "one point quantity" and results from the difference between g, i.e. the magnitude of $\boldsymbol{g} = \boldsymbol{grad}\, W$, and γ at the same point of the boundary, cf. (Koch and Pope, 1972), (Bjerhammar and Svensson, 1983) and (Grafarend, 1989).

Returning now to the weak solution concept, we immediately see that the approach applied to Neumann's problem needs some modifications so as to express the oblique derivative boundary condition given by Eq. (32). In particular this concerns the structure of the bilinear form $A(u,v)$. In the sequel, therefore, in contrast to Eq. (26) we put

$$A(u,v) = A_1(u,v) - A_2(u,v), \tag{33}$$

where

$$A_1(u,v) = \int_\Omega \left\langle \boldsymbol{grad}\, u, \boldsymbol{grad}\, v \right\rangle d\boldsymbol{x}, \tag{34}$$

$$A_2(u,v) = \int_\Omega \left\langle \boldsymbol{grad}\, v, \boldsymbol{a} \times \boldsymbol{grad}\, u \right\rangle d\boldsymbol{x} + \int_\Omega v \left\langle \boldsymbol{curl}\, \boldsymbol{a}, \boldsymbol{grad}\, u \right\rangle d\boldsymbol{x} \tag{35}$$

and $\boldsymbol{a} = (a_1, a_2, a_3)$ is a vector field such that a_i and also $|\boldsymbol{x}|\,(\boldsymbol{curl}\, \boldsymbol{a})_i$, $i = 1,2,3$, are Lebesgue measurable functions defined and bounded almost everywhere on Ω. Moreover, we assume that on the boundary $\partial\Omega$ the vector $\boldsymbol{\sigma} = \boldsymbol{s} / \left\langle \boldsymbol{s}, \boldsymbol{n} \right\rangle$ and the field $\boldsymbol{a}$ are coupled so that $\boldsymbol{\sigma} = \boldsymbol{n} + \boldsymbol{a} \times \boldsymbol{n}$. Note also that the tie to the classical formulation of the problem requires that $f = \gamma\left(\partial U/\partial n\right)^{-1}\delta g$, see (Holota, 1997, 2000, 2005a,b).

The structure of $A(u,v)$ is rather complex. Nevertheless, the solution of our oblique derivative problem can be approached in terms of successive approximations. Indeed, we can construct a sequence of functions T_m, $m = 0,1,\dots,\infty$, defined by the following equations (integral identities)

$$A_1(T_{m+1},v) = \int_{\partial\Omega} vf \, dS + A_2(T_m,v) \tag{36}$$

which are assumed to hold for all $v \in W_2^{(1)}(\Omega)$ and $m = 0,1,\dots,\infty$. Subsequently, under some limitations, we can even show that $[T_m]_{m=0}^{\infty}$ is a Cauchy sequence in $W_2^{(1)}(\Omega)$ that in the norm of the Sobolev space $W_2^{(1)}(\Omega)$ converges to the solution of our weakly formulated oblique derivative problem, see (Holota, 2000).

Under certain regularity assumptions the iterations may be even interpreted as follow

$$A_1(T_{m+1},v) = \int_{\partial\Omega} vf_m \, dS \tag{37}$$

valid for all $v \in W_2^{(1)}(\Omega)$, while

$$f_m = f - \frac{\partial T_m}{\partial t} \tan(\boldsymbol{s},\boldsymbol{n}) \tag{38}$$

with $\partial/\partial t$ denoting the derivative in the direction of $\boldsymbol{t} = (\boldsymbol{\sigma} - \boldsymbol{n})/|\boldsymbol{\sigma} - \boldsymbol{n}|$, which obviously is tangential to $\partial\Omega$ (and exists almost everywhere on Lipschitz' boundary $\partial\Omega$), see (Holota, 2000).

10. Linear system

Recall that in our case we are looking for the solution of Laplace's partial differential equation. It is, therefore, enough to consider just the space $H_2^{(1)}(\Omega)$ of those functions from $W_2^{(1)}(\Omega)$ which are harmonic in Ω and to reformulate our problem, i.e. to look for $T_{m+1} \in H_2^{(1)}(\Omega)$ such that

$$A_1(T_{m+1}, v) = \int_{\partial\Omega} v f_m \, dS \tag{39}$$

holds for all $v \in H_2^{(1)}(\Omega)$. Subsequently we can approximate T_{m+1} by means of

$$T^{(n,m)} = \sum\nolimits_{j=1}^{n} c_j^{(n,m)} v_j \tag{40}$$

where v_j are members of a function basis of $H_2^{(1)}(\Omega)$. The coefficients $c_j^{(n,m)}$ then result from the solution of Galerkin's system

$$\sum\nolimits_{j=1}^{n} c_j^{(n,m)} A_1(v_j, v_k) = \int_{\partial\Omega} v_k f_m \, dS, \; k = 1, \ldots, n \tag{41}$$

Here definitely practical aspects come into play since the computation of all the entries $A_1(v_j, v_k)$ of Galerkin's matrix, especially for high *n*, is rather demanding, even for high performance computation facilities. In practice, therefore, it is useful to put

$$A^*(u, v) = \int_{\Omega^*} \langle \boldsymbol{grad}\, u, \boldsymbol{grad}\, v \rangle \, d\boldsymbol{x}, \text{ for } u, v \in H_2^{(1)}(\Omega^*), \tag{42}$$

where Ω^* has a "simpler boundary". However, for v_j we have to take members of a function basis in $H_2^{(1)}(\Omega^*)$. In $H_2^{(1)}(\Omega^*)$ they generate a sequence of finite dimensional subspaces $H_n(\Omega^*) = \mathrm{span}\{v_i, i = 1, \ldots, n\}$, $n = 1, 2, \ldots$ We will suppose that $\Omega \subseteq \Omega^*$. Thus, using Runge's property of Laplace's equation, see e.g. (Bers et al., 1964), (Krarup, 1969) or (Moritz, 1980), we can see that v_j (restricted to Ω) form a function basis in $H_2^{(1)}(\Omega)$ too. By analogy then $H_n(\Omega) = \mathrm{span}\{v_i, i = 1, \ldots, n\}$, $n = 1, 2, \ldots$ and it is clear that $H_n(\Omega) = H_n(\Omega^*)\big|_{\Omega}$. Thus for all $v \in H_n(\Omega^*)$

$$A^*(u^{(n,m)}, v) = \int_{\partial\Omega} v f_m \, dS + F(v), \tag{43}$$

where

$$F(v) = A^*(u^{(n,m)}, v) - A_1(u^{(n,m)}, v) = \int_D \left\langle \boldsymbol{grad}\, u^{(n,m)}, \boldsymbol{grad}\, v \right\rangle d\boldsymbol{x} \tag{44}$$

and $D = \Omega^* - \Omega$. Note that the equations above imply a continuation of $u^{(n,m)}$ and that Eq.(43) is an alternative expression of the original integral identity. Clearly, the practical solution technique then rests on successive approximations. The problems were investigated and also subjected to extensive numerical tests in (Holota and Nesvadba, 2007a, 2012a) and (Nesvadba et al., 2007).

11. Reproducing kernel and Galerkin's system

The function space $H_2^{(1)}(\Omega^*)$ considered in the last section is a Hilbert space equipped with scalar product defined by the bilinear form $A^*(u,v)$. It is not extremely difficult to show that with respect to this scalar product $H_2^{(1)}(\Omega^*)$ is a reproducing Hilbert space, i.e., there exists a kernel $K(\boldsymbol{x},\boldsymbol{y})$ which is an element of $H_2^{(1)}(\Omega^*)$ such that

$$\sum_{i=1}^{3} \int_{\Omega^*} \frac{\partial K(\boldsymbol{x},\boldsymbol{y})}{\partial x_i} \frac{\partial v(\boldsymbol{x})}{\partial x_i}\, d\boldsymbol{x} = v(\boldsymbol{y}) \text{ holds for all } v \in H_2^{(1)}(\Omega^*). \tag{45}$$

Note that the existence of $K(\boldsymbol{x},\boldsymbol{y})$ with the reproducing property is not a matter of course. For instance in Sobolev's space $W_2^{(1)}(\Omega)$ there is no kernel of this kind, which can be deduced from Sobolev's lemma on embedding.

In case of some simple domains the kernel which has the reproducing property in $H_2^{(1)}(\Omega^*)$ can be explicitly constructed. For instance in case that Ω^* is the exterior of a sphere of radius $R > 0$ we have

$$K(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi R} \sum_{n=0}^{\infty} \frac{2n+1}{n+1} \rho^{n+1} P_n(\cos\psi), \text{ where } \rho = \frac{R^2}{|\boldsymbol{x}||\boldsymbol{y}|}, \tag{46}$$

P_n is Legendre's function of the first kind and Ψ is the angel between the position vectors of the computation and the moving point of the integration. In addition it is not extremely difficult to find that

$$K(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi R} \left(\frac{2\rho}{L} - \ln \frac{L + \rho - \cos\psi}{1 - \cos\psi} \right), \tag{47}$$

where $L = \sqrt{1 - 2\rho\cos\psi + \rho^2}$, see e.g. (Tscherning, 1975), (Neyman, 1979), (Holota, 2004, 2011).

For approximation purposes the existence of the reproducing kernel in the particular Hilbert space is extremely useful. Indeed, supposing that $\boldsymbol{y}_i \in \Omega^*$, $i = 1,\ldots,\infty$ is a sequence of points which is dense in Ω^* then the linear manifold $H = \operatorname{span}\{K(\boldsymbol{x},\boldsymbol{y}_i), i = 1,\ldots,\infty\}$ is densely embedded in $H_2^{(1)}(\Omega^*)$, see (Sansò, 1986). Hence $K(\boldsymbol{x},\boldsymbol{y})$ gives us a possibility to generate finite dimensional subspaces $H_n = \operatorname{span}\{K(\boldsymbol{x},\boldsymbol{y}_i), i = 1,\ldots,n\}$ in $H_2^{(1)}(\Omega^*)$ such that $H_n \subseteq H_{n+1}$ and $\lim_{n\to\infty} \operatorname{dist}(v, H_n) = 0$ for all $v \in H_2^{(1)}$, i.e., $\lim_{n\to\infty} H_n = H_2^{(1)}$. This is important since for $v_j(\boldsymbol{x}) = K(\boldsymbol{x},\boldsymbol{y}_j)$ the elements $A^*(v_j, v_k)$ in Galerkin's matrix may be immediately expressed by

$$A^*(v_j, v_k) = K(\boldsymbol{y}_j, \boldsymbol{y}_k) \tag{48}$$

in view of the reproducing property of the kernel.

12. Reproducing kernel for an ellipsoid

The possibility to express $A^*(v_j, v_k)$ by means of $K(\boldsymbol{y}_j, \boldsymbol{y}_k)$ is a strong stimulus for constructing the reproducing kernel also in case that Ω^* is the exterior Ω_{ell} of an oblate ellipsoid of revolution of semi-axes a and b, $a \geq b$. In comparison with the sphere Ω_{ell} is much closer to the real solution domain Ω. This has very positive effect on the convergence of the iteration procedure applied for the solution of the gravimetric boundary value problem under consideration. A short note motivating interest in ellipsoidal reproducing kernels can be also found in (Tscherning, 2004). In the sequel we will denote the kernel by $K_{ell}(\boldsymbol{x},\boldsymbol{y})$.

Naturally, we will use ellipsoidal coordinates u, β, λ. They are related to x_1, x_2, x_3 by the equations

$$x_1 = \sqrt{u^2 + E^2}\cos\beta\cos\lambda,\ x_2 = \sqrt{u^2 + E^2}\cos\beta\sin\lambda,\ x_3 = u\sin\beta, \tag{49}$$

where $E = \sqrt{a^2 - b^2}$. Note that in the coordinates u, β, λ the boundary $\partial\Omega_{ell}$ of Ω_{ell} is defined by $u = b$. Starting now from the reproducing property of the kernel represented by Eq. (45) and referring to (Holota, 2004 and 2011), we can deduce that

$$K_{ell}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi b}\sum_{n=0}^{\infty}(2n+1)\Big[K_{n0xy}P_n(\sin\beta_x)P_n(\sin\beta_y) + + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}K_{nmxy}\,P_{nm}(\sin\beta_x)P_{nm}(\sin\beta_y)\cos m(\lambda_x-\lambda_y)\Big], \tag{50}$$

where

$$K_{nmxy} = \frac{iEb}{a^2}\frac{Q_{nm}(z_x)}{Q_{nm}(z_0)}\frac{Q_{nm}(z_y)}{Q_{nm}(z_0)}\left[\frac{dQ_{nm}(z_0)}{dz}\Big/Q_{nm}(z_0)\right]^{-1}, \tag{51}$$

P_{nm} and Q_{nm} are Legendre's functions of the 1st and the 2nd kind, while

$$z_x = \frac{iu_x}{E},\ z_y = \frac{iu_y}{E},\ z_0 = \frac{ib}{E} \text{ and } i=\sqrt{-1}. \tag{52}$$

Unfortunately, the numerical implementation of $K_{ell}(\boldsymbol{x},\boldsymbol{y})$ on the basis of Eqs. (50) and (51) is extremely demanding, especially when all the entries of Galerkin's matrix and right sides in Galerkin's system have to be computed for high resolution modelling of the solution. This motivates studies leading to an (analytical) summation of the series that represents the kernel. Under some approximations discussed in details in (Holota, 2014), we arrive at $\tilde{K}_{ell}(\boldsymbol{x},\boldsymbol{y})$ that approximates $K_{ell}(\boldsymbol{x},\boldsymbol{y})$ with a relatively high degree of accuracy and is given by

$$\tilde{K}_{ell}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi b}K^{(1)}(\boldsymbol{x},\boldsymbol{y}) - \frac{E^2}{4\pi ab^2}K^{(2)}(\boldsymbol{x},\boldsymbol{y}) + \frac{E^2}{4\pi ab^2}K^{(3)}(\boldsymbol{x},\boldsymbol{y}), \tag{53}$$

where

$$K^{(1)}(\boldsymbol{x},\boldsymbol{y}) = \sum_{n=0}^{\infty}\frac{2n+1}{n+1}\rho^{n+1}P_n(\cos\psi), \tag{54}$$

$$K^{(2)}(\boldsymbol{x},\boldsymbol{y}) = \sum_{n=0}^{\infty}\frac{2n+1}{2n+3}\rho^{n+1}P_n(\cos\psi), \tag{55}$$

$$K^{(3)}(\boldsymbol{x},\boldsymbol{y}) = -\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2(2n+3)}\,\rho^{n+1}\,\frac{\partial^2 P_n(\cos\psi)}{\partial\lambda^2} \tag{56}$$

and Ψ denotes the angular distance of points (β_x, λ_y) and (β_y, λ_y) on the sphere, when β and λ are interpreted as spherical latitude and longitude, respectively. Of course, the problem is the summation of the series on the righ hand side of Eqs. (54) - (56). In (Holota and Nesvadba, 2014) we can find that

$$K^{(1)}(\boldsymbol{x},\boldsymbol{y}) = \frac{2\rho}{L} - \ln\frac{L+\rho-\cos\psi}{1-\cos\psi}, \tag{57}$$

cf. Eq. (47),

$$K^{(2)}(\boldsymbol{x},\boldsymbol{y}) = \frac{\rho}{L} - 2\sqrt{1-k^2\sin^2\varphi} - \left(\tan\frac{\varphi}{2}\right)^{-1}\left[\mathrm{F}\,(k,\varphi) - 2\mathrm{E}(k,\varphi)\right], \tag{58}$$

where $\varphi \in \langle 0, \pi/2 \rangle$ is a new variable whose relation to ρ is given by

$$\rho = \tan^2\frac{\varphi}{2}, \tag{59}$$

$$k^2 = \cos^2\frac{\psi}{2}, \tag{60}$$

and

$$\mathrm{F}\,(k,\varphi) = \int_0^{\varphi} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} \quad \text{and} \quad \mathrm{E}(k,\varphi) = \int_0^{\varphi} \sqrt{1-k^2\sin^2\varphi}\, d\varphi \tag{61}$$

are the Legendre (incomplete) elliptic integrals of the first and the second kind. Finally, for $K^{(3)}(\boldsymbol{x},\boldsymbol{y})$, referring to (Holota and Nesvadba, 2014) again, we have

$$K^{(3)}(\boldsymbol{x},\boldsymbol{y}) = S_1 \sin^2\alpha_{xy} \cos^2\beta_y - \mu S_2, \tag{62}$$

where α_{xy} is the azimuth of the point (β_y, λ_y) as seen from the point (β_x, λ_y), when β and λ are interpreted as spherical latitude and longitude, respectively,

$$\mu = \sin\beta_x \sin\beta_y - \left(1 - 2\sin^2\alpha_{xy}\cos^2\beta_y\right)\cos\psi\,, \tag{63}$$

while

$$\begin{aligned} S_1 &= \frac{\rho}{L} + \ln\frac{L+\rho-\cos\psi}{1-\cos\psi} - 6\sqrt{1-k^2\sin^2\varphi} - \\ &\quad - 3\left(\tan\frac{\varphi}{2}\right)^{-1}\left[\mathrm{F}\,(k,\varphi) - 2\mathrm{E}(k,\varphi)\right], \end{aligned} \tag{64}$$

$$S_2 = \frac{1}{\sin^2\psi}\left[S_{21}\cos\psi - S_{22}\right], \tag{65}$$

$$\begin{aligned} S_{21} &= -\frac{\rho}{3} - \ln\frac{L+\rho-\cos\psi}{1-\cos\psi} + 4\sqrt{1-k^2\sin^2\varphi} + \\ &\quad + 2\left(\tan\frac{\varphi}{2}\right)^{-1}\left[\mathrm{F}\,(k,\varphi) - 2\mathrm{E}(k,\varphi)\right], \end{aligned} \tag{66}$$

and

$$S_{22} = -4 - \frac{\rho}{3}\cos\psi + \ln\frac{L+1-\rho\cos\psi}{2} + 2\left(\tan\frac{\varphi}{2}\right)^{-1}\mathrm{F}\,(k,\varphi)\,. \tag{67}$$

The numerical tests discussed in (Holota and Nesvadba, 2014) show that $\tilde{K}_{ell}(\boldsymbol{x},\boldsymbol{y})$ may serve as an efficient tool for solving potential problems in gravity field studies, in particular for constructing Galerkin's approximations of the disturbing potential.

13. Concluding remarks

The historical relation between geodesy and mathematics is well-known. The role of potential theory in gravity field studies represents one of its important forms. In a sense this was also demonstrated in our discussion on the use of the theory of boundary value problems in the determination of the gravitational potential and figure of the Earth. Nevertheless, the real picture of the situation is even more varied and shows aspects that have to be taken into consideration in the solution of our problem. This became particularly apparent when satellite data came into play.

We saw it, for instance, in our reasoning justifying the transition from the free to fixed boundary value problem in the determination of the gravitational potential. Another example is the use of mixed boundary value problems that may serve as a mathematical model for gravity field studies based on heterogeneous boundary date, see e.g. (Holota, 1982, 1983a,b,c).

In addition it is also necessary to take into consideration the fact that in the majority of cases the determination of the gravitational field from the combination of terrestrial and satellite data represents an overdetermined problem. Hence, an optimization approach has to be applied together with the concept of boundary value problems. This is another rich field that deserves attention. Some aspects and tools concerning this topic are discussed in (Holota, 2007, 2009), (Holota and Kern, 2005), (Holota and Nesvadba, 2007b, 2009, 2012b).

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References

Bers L., John F. and Schechter M. (1964). Partial differential equations. J Wiley and Sons, Inc., New York–London–Sydney.

Bjerhammar A. and Svensson L. (1983). On the geodetic boundary-value problem for a fixed boundary surface – A satellite approach. Bull. Géod. 57, 382-393.

Giaquinta M. (2000). Hilbert e il calcolo delle variazioni. Le Matematiche, Vol. LV, Supplemento n. 1, 47-58.

Grafarend E.W. (1989). The geoid and the gravimetric boundary-value problem. Report No. 18 from the Dept. of Geod., The Royal Inst. of Technology, Stockholm.

Heiskanen W. and Moritz H. (1967). Physical Geodesy. W.H. Freeman and Company, San Francisco and London.

Holota P. (1977). Determination of the free boundary in potential theory, the boundary problem of physical geodesy. Proc. 3rd Int. Symp. Geod. and Phys. of the Earth. GDR, Weimar, 25-31 October 1976. Veröff. d. Zentralinst. f. Phys. d. Erde, No. 52, Teil 2, 269-279.

Holota P. (1980). The boundary problem of physical geodesy after the appearance of Lars Hormander's paper: Present State. Proc. of the Summer School of KAPG working group 5.2 "The modern potential theory as a basis for the solution of geo-

physical inverse problems", Freiberg, 1978. Geod. Geoph. Veroff. R. III, H. 45, 182-198.
Holota P. (1982). The altimetry-gravimetry boundary value problem. Proc. Int. Conf. Intercosmos-Section 6, Albena, 1980. Obs. of Artif. Earth Sat., No. 20, Sofia, 234-249.
Holota P. (1983a). The altimetry-gravimetry boundary value problem I: Linearization, Friedrichs' inequality. Proc. 8th Hotine symp. on Math. Geod., Como, 1981. Anno XLII – Boll. di Geod. e Sci. Affini – N. 1, 13-32.
Holota P. (1983b). The altimetry-gravimetry boundary value problem II: Weak solution, V-ellipticity. Proc. 8th Hotine symp. on Math. Geod., Como, 1981. Anno XLII – Boll. di Geod. e Sci. Affini – N. 1, 69-84.
Holota P. (1983c). Mixed boundary value problems in physical geodesy. In: Holota P. (ed.): Proc. Int. Symp. Figure of the Earth, the Moon and other Planets, Prague, Sept. 20-25, 1982. Monograph Series of the Research Inst. of Geod., Topogr. and Cartogr., Prague, 253-285.
Holota P. (1985). A new approach to iteration solutions in solving geodetic boundary value problems for real topography. Proc. 5th Int. Symp. Geod. and Phys. of the Earth, GDR, Magdeburg, Sept. 23rd-29th, 1984, Part II. Veroff. d. Zentr. Inst. f. Phys. d. Erde, Nr. 81, Teil II, 4-15.
Holota P. (1989). Laplacian versus topography in the solution of the Molodensky problem by means of successive approximations. In: Kejlso E., Poder K. and Tscherning C.C. (eds.): Festschrift to Torben Krarup. Geodaetisk Inst., Meddelelse No. 58, Kobenhavn, 213-227.
Holota P. (1992a). On the iteration solution of the geodetic boundary-value problem and some model refinements. Contribution to Geodetic Theory and Methodology. XXth General Assembly of the IUGG, IAG-Sect. IV, Vienna, 1991. Politecnico di Milano, 1991, 31-60, also in: Travaux de l'Association Internationale de Geodesie, Tome 29, Paris, 260-289.
Holota P. (1992b). Integral representation of the disturbing potential: Effects involved, iteration technique and its convergence. In: Holota P. and Vermeer M. (eds.): Proc. First continental workshop on the geoid in Europe, Prague, May 11-14, 1992. Research Inst. of Geod., Topog. and Cartog., Prague, in co-operation with IAG-Subcommis. for the Geoid in Europe, Prague, 402-419.
Holota P. (1995). Classical methods for non-spherical boundary problems in physical geodesy. In: Sanso F. (ed.): Geodetic Theory Today, International Association of Geodesy Symposia, Vol. 114, Springer, Berlin – Heidelberg – New York, 315-324.
Holota P. (1997). Coerciveness of the linear gravimetric boundary-value problem and a geometrical interpretation. Journal of Geodesy, 71, 640-651.
Holota P. (1999). Variational methods in geoid determination. and function bases. Physics and Chemistry of the Earth, Part A: Solid Earth and Geodesy, Vol 24, No 1, 3-14.

Holota P. (2000). Direct method in physical geodesy. In: Schwarz K.P. (ed.): Geodesy Beyond 2000, The Challenges of the First Decade. IAG Symposia, Vol. 121, Springer-Verlag, 163-170.

Holota P. (2003). Green's function and external masses in the solution of geodetic boundary-value problems. In: Tziavos I.N. (ed.): Gravity and Geoid, 3rd Meeting of the Intl. Gravity and Geoid Commission, Thessaloniki, Greece, August 26-30, 2002. Ziti Editions, Thessaloniki, 108-113.

Holota P. (2004). Some topics related to the solution of boundary-value problems in geodesy. In: Sansò F. (ed.): V Hotine-Marussi Symposium on mathematical Geodesy, Matera, Italy, June 17-21, 2002. Intl. Assoc. of Geodesy Symposia, Vol. 127, Springer, Berlin – Heidelberg – New York, 2004, 189-200.

Holota P. (2005a). Successive approximations in the solution of weakly formulated geodetic boundary-value problem. In: Sansò F. (ed.): A Window on the Future of Geodesy. IAG General Assembly, Sapporo, Japan, June 30 - July 11, 2003. Intl. Assoc. of Geodesy Symposia, Vol. 128, Springer, Heidelberg – New York, 452-458.

Holota P. (2005b). Neumann's boundary-value problem in studies on Earth gravity field: Weak solution. In: Holota P. and Slaboch V. (eds.): 50 years of the Research Institute of Geodesy, Topography and Cartography, Jubilee Proceedings 1954-2004. Res. Inst. of Geod., Topogr. and Cartogr, Zdiby-Prague, Vol. 50, Publication No. 36, 2005, 49-69.

Holota P. (2007): On the combination of terrestrial gravity data with satellite gradiometry and airborne gravimetry treated in terms of boundary-value problems. In: Tregoning P. and Rizos C. (eds.): Dynamic Planet – Monitoring and Understanding a Dynamic Planet with Geodetic and Oceanographic Tools. IAG Symposium, Cairns, Australia, 22.-26. August 2005. International Association of Geodesy Symposia, Vol. 130, Springer, Berlin – Heidelberg – New York, Chap. 53, 362-369.

Holota P. (2009). On the target function in combining terrestrial and satellite gravity field data. In: Holota P. (ed.): Mission and Passion: Science. A volume dedicated to Milan Burša on the occasion of his 80th birthday. Published by the Czech National Committee of Geodesy and Geophysics. Prague, 101-112.

Holota P. (2011). Reproducing kernel and Galerkin's matrix for the exterior of an ellipsoid: Application in gravity field studies. Studia geophysica et geodaetica, 55(2011), No. 3, 397-413.

Holota P. and Kern M. (2005). A study on two-boundary problems in airborne gravimetry and satellite gradiometry. In: Jekeli C., Bastos L. and Fernandes J. (eds.): Gravity, Geoid and Space Missions. GGSM 2004, IAG International Symposium, Porto, Portugal, August 30 - September 3, 2004. International Association of Geodesy Symposia, Vol. 129, Springer, Berlin – Heidelberg – New York, 173-178..

Holota P. and Nesvadba O. (2007a). Model refinements and numerical solutions of weakly formulated boundary-value problems in physical geodesy. In: Xu P., Liu J.

and Dermanis A. (eds.): VI Hotine-Marussi Symposium on Theoretical and Computational Geodesy. International Association of Geodesy Symposia, Vol. 132, Springer, Berlin – Heidelberg, 320-326.

Holota P. and Nesvadba O. (2007b). Optimized solution and a numerical treatment of two-boundary problems in combining terrestrial and satellite data. HARITA DERGISI, Proceedings of the 1st International Symposium of the International Gravity Field Service "Gravity Field of the Earth", 28 August – 1 September, 2006, Istanbul, Turkey. Special Issue: 18, General Command of Mapping, Ankara, 25-30.

Holota P. and Nesvadba O. (2009). Domain transformation, boundary problems and optimization concepts in the combination of terrestrial and satellite gravity field data. In: Sideris M. (ed.): Observing our Changing Earth, Proceedings of the 2007 IAG General Assembly, Perugia, Italy, July 2 - 13, 2007. International Association of Geodesy Symposia, Vol., Springer, Berlin – Heidelberg, 219-228.

Holota P. and Nesvadba O. (2012a). Method of successive approximations in solving geodetic boundary value problems: analysis and numerical experiments. In: Sneeuw N., Novak P., Crespi M. and Sansò F. (eds.): VII Hotine-Marussi Symposium on Mathematical Geodesy. Proceedings of the Symposium in Rome, 6-10 June 2009. International Association of Geodesy Symposia, Vol. 137, Springer-Verlag, Berlin – Heidelberg, 189-198.

Holota P. and Nesvadba O. (2012b). On a combined use of satellite and terrestrial data in refined studies on Earth gravity field: Boundary problems and a target function. In: Kenyon S.C., Pacino M.C., and Urs J.M. (eds.): Geodesy for Planet Earth. Proceedings of the 2009 IAG Symposium, Buenos Aires, Argentina, 31 August - 4 September 2009. International Association of Geodesy Symposia, Vol. 136, Springer-Verlag, Berlin – Heidelberg, 195-204.

Holota P. and Nesvadba O. (2014). Reproducing kernel and Neumann's function for the exterior of an oblate ellipsoid of revolution: Application in gravity field studies. Studia geophysica et geodaetica (to appear).

Hörmander L. (1976). The boundary problems of physical geodesy. Archive for Rational Mechanics and Analysis, Vol. 62, No. 1. Springer Verlag, Berlin – Heidelberg – New York.

Kellogg O.D. (1953). Foundations of potential theory. Dover Publications, Inc., New York.

Koch K.R. and Pope A.J. (1972). Uniqueness and existence for the geodetic boundary-value problem using the known surface of the Earth. Bull. Géod. 106, 467-476.

Krarup T. (1969). A contribution to the mathematical foundation of physical geodesy. Danish Geod. Inst., Publ. No. 44, Copenhagen.

Krarup T. (1973). Letters on Molodensky's problem I-IV, Communication to the members of IAG Special Study Group 4.31.

Kufner A., John O. and Fucik S. (1977). Function spaces. Academia, Prague.

Mikhlin S.G. (1962). Multidimensional singular integrals and integral equations. State publishers of physical and mathematical literature, Moscow (in Russian).
Michlin S.G. (1970). Variational methods in mathematical physics. Nauka Publisher, Moscow (in Russian); also in Slovakian: Alpha, Bratislava, 1974 and in English: Pergamon Press, New York, 1964.
Molodensky M.S., Eremeev V.F. and Yurkina M.I. (1962). Methods for study of the external gravitational field and figure of the Earth. Transl. from Russian (1960). Israel Program for Scientific Translations, Jerusalem; Russian original published in Trudy CNIIGAiK, Vol. 131, Geodezizdat, Moscow, 1960.
Moritz H. (1972). Convergence of Molodensky's series. Report No. 183, Department of Geodetic Science, The Ohio State University, Columbus.
Moritz H. (1973). On the convergence of Molodensky's series. Boll. Geod. Sci. Affini, 32, 125-144.
Moritz H. (1977). Recent developments in the geodetic boundary-value problem. Report No. 266, Department of Geodetic Science, The Ohio State University, Columbus.
Moritz H. (1980). Advanced Physical Geodesy. Herbert Wichmann Verlag, Karlsruhe and Abacus Press, Tunbridge, Wells Kent.
Necas J. (1967). Les méthodes directes en théorie des équations elliptiques. Academia, Prague.
Necas J. and Hlavacek I. (1981). Mathematical theory of elastic and elasto-plastic bodies: An introduction. Elsevier Sci. Publ. Company, Amterdam – Oxford – New York.
Nesvadba O., Holota P. and Klees R. (2007). A direct method and its numerical interpretation in the determination of the Earth's gravity field from terrestrial data. In: Tregoning P. and Rizos C. (eds.): Dynamic Planet – Monitoring and Understanding a Dynamic Planet with Geodetic and Oceanographic Tools. IAG Symposium, Cairns, Australia, 22.-26. August 2005. International Association of Geodesy Symposia, Vol. 130, Springer, Berlin – Heidelberg – New York, Chap. 54, 370-376.
Neyman Yu. (1979). A variational method of physical geodesy. Nedra Publishers, Moscow (in Russian).
Rektorys K. (1974). Variacni metody v inzenyrskych problemech a v problemech matematicke fyziky. SNTL Publishers of Technical Literature, Prague 1974; also in English: Variational methods. Reidel Co., Dordrecht–Boston, 1977.
Sansò F. (1977). The geodetic boundary value problem in gravity space. Memorie, Accademia Nazionale dei Lincei, ser. VIII, Vol. XIV, 3, 39-97, Roma.
Sansò F. (1978a). Molodensky's problem in gravity space: a review of the first results. Bull. Géod., No. 52, 59-70.
Sansò F. (1978b). The local solvability of Molodensky's problem in gravity space. Manuscripta geodaetica, No. 3, 157-227.

Sansò F. (1986). Statistical methods in physical geodesy. In: Sünkel H. (ed.): Mathematical and numerical techniques in physical geodesy. Lecture Notes in Earth Sciences, Vol. 7, Springer-Verlag, Berlin–Heidelberg–New York etc., 49-155.

Tscherning C.C. (1975). Application of collocation. Determination of a local approximation to the anomalous potential of the Earth using "exact" astro-gravimetric collocation. In: Bosowski B. and Martensen E. (eds.): Methoden und Verfaren der Mathematischen Physik, Vol. 14. 83-110.

Tscherning C.C. (2004). A Discussion on the Use of Spherical Approximation or no Approximation in Gravity Field Modelling with Emphasis on Unsolved Problems in Least-Squares Collocation. In: Sansò F. (ed.), *V Hotine-Marussi Symposium on Mathematical Geodesy*, Matera, Italy, June 17-21, 2002. International Association of Geodesy Symposia, 127. Springer, Berlin – Heidelberg, 184-188.