Sitzungsberichte der Leibniz-Sozietät der Wissenschaften zu Berlin *119(2014), 55–74*

Fernando Sanso

The boundary elements formulation of Molodensky's problem: new ideas from the old book Physical Geodesy

This paper is based on an idea born in reading once more Physical Geodesy and it is dedicated to Helmut Moritz who has been my teacher of Geodesy

Abstract

After a short recall of the definition of Molodensky's problem and of its analysis, from early times to more modern papers, the attention is focussed on the linearized version of the problem, in its scalar form; this is then suitably reformulated into the so-called planar approximation. The primary goal of the problem is that one would like to retrieve the anomalous gravity potential T on the surface $\widetilde{S}$, called telluroid, that approximates the Earth surface S, from the free air gravity anomalies Δg given on $\widetilde{S}$. Would Δg be related simply to the normal derivative of T on $\widetilde{S}$, a suitable use of the third Green identity allowed to write a (regular) integral equation, the solution of which gives T on $\widetilde{S}$. This approach is known in mathematics as the Boundary Elements Method (BEM). However Δg is rather related to a directional derivative of T, oblique with respect to $\widetilde{S}$. Also in this case the third Green identity can be put in the form of an integral equation, which this time has to be strongly singular. this is usually done by exploiting a coordinate system adapted to the normal to $\widetilde{S}$ and a couple of tangential coordinates.

Yet in the present setting by exploiting a different decomposition of the normal of $\widetilde{S}$ one can reach the above target in a simpler manner. In this way a suitable BEM integral equation is formulated in this paper. Such an equation is known to enjoy the Fredholm alternative property, so that proving the uniqueness of the solution of the problem is of a great importance. this is done by adapting to the present Euclidean setting the proofs already present in literature.

The nice feature of the theorem is that one has to put on $\widetilde{S}$ the only requirement to have a finite inclination ($< 90°$) with respect to the horizontal (x, y) plane.

1 Formulation of the problem

The Scalar Molodensky problem is to find the gravity potential $W(\mathbf{x})$ in the outer space and the surface S, at all points P of which we know

$P \in S$;	(λ_P, φ_P)	ellipsoidal coordinates of P
	$W(\lambda_P, \varphi_P)$	the potential on S
	$g(\lambda_P, \varphi_P)$	the modulus of the gravity vector on S

If we stipulate that the unknown S can be described by the vector function

$$\mathbf{x}(\lambda,\varphi) = \mathbf{x}_e(\lambda,\varphi) + h(\lambda,\varphi)\boldsymbol{\nu}(\lambda,\varphi), \quad (1)$$
$$P \equiv \{\mathbf{x}(\lambda,\varphi)\}$$

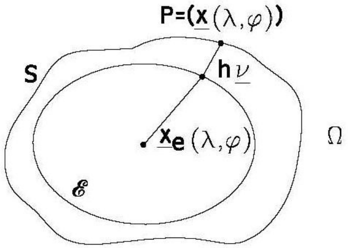

Figure 1: The surface S is in one to one normal correspondence to the ellipsoid $\mathcal{E}$.

where $\boldsymbol{\nu}(\lambda,\varphi)$ is the normal to the ellipsoid in the direction of P, and if we recall that

$$W(\mathbf{x}) = V(\mathbf{x}) + \frac{1}{2}\omega^2(x^2+y^2) \quad (2)$$

with $V(\mathbf{x})$ the purely Newtonian component of W, that is harmonic outside S, the problem can be conveniently formulated as

$$\Delta\{W - \frac{1}{2}\omega^2(x^2+y^2)\} = 0 \ \text{ in } \Omega \quad (3)$$

$$W|_S = w(\lambda,\varphi) \ \text{ on } S \quad (4)$$

$$|\nabla W||_S = g(\lambda,\varphi) \ \text{ on } S \quad (5)$$

This is a non-linear, free-boundary, oblique derivative boundary value problem for the Laplace equation.

This formulation and its first analysis in Hölder spaces can be found in [18].

Since the normal gravity potential $U(\mathbf{x})$ approximates $W(\mathbf{x})$ with a relative error of the order of 10^{-5}, it is natural that (3), (4), (5) can be linearized by setting

$$W(\mathbf{x}) = U(\mathbf{x}) + T(\mathbf{x}) \tag{6}$$

and expressing a new linear problem for the anomalous potential T.

If we define a telluroid $\widetilde{S}$ as

$$\widetilde{S} \equiv \{\widetilde{\mathbf{x}}(\lambda,\varphi) = \mathbf{x}_e(\lambda,\varphi) + \widetilde{h}(\lambda,\varphi)\boldsymbol{\nu}(\lambda,\varphi)\} \tag{7}$$

where $\widetilde{h}$, the so-called normal height, is given by the solution of the equation

$$U[\mathbf{x}_e(\lambda,\varphi) + \widetilde{h}(\lambda,\varphi)\boldsymbol{\nu}(\lambda,\varphi)] = w(\lambda,\varphi) \equiv W\Big|_S \ , \tag{8}$$

then one easily proves that [7, 21]

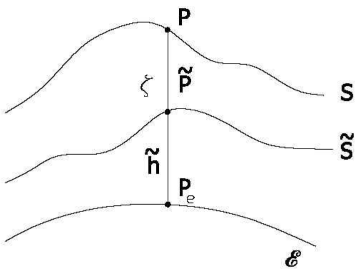

Figure 2: The geometry of the telluroid and height anomaly ζ

in a linear approximation

$$\zeta(\lambda,\varphi) = \frac{T(\widetilde{T})}{\gamma(\widetilde{P})} \tag{9}$$

where $\gamma(P) = |\boldsymbol{\gamma}(P)| = |\nabla U(P)|$, and

$$\Delta g(\lambda,\varphi) = g(P) - \gamma(\widetilde{P}) = -\frac{\partial T}{\partial h} + \frac{\frac{\partial \gamma}{\partial h}}{\gamma} T\Big|_{\widetilde{S}} \ . \tag{10}$$

Therefore our problem (3), (4), (5) can now be formulate as:

find T solution of the oblique derivative BVP

$$\begin{cases} \Delta T = 0 & \text{in } \widetilde{\Omega} \equiv \{h \geq \widetilde{h}\} \\ -\frac{\partial T}{\partial h} + \frac{\frac{\partial \gamma}{\partial h}}{\gamma} T \Big|_{\widetilde{S}} = \Delta g(\lambda, \varphi) & \text{on } \widetilde{S} \end{cases} \tag{11}$$

In this form the problem dates back to the work of [15]. Note has to be taken that (11) is not complemented by asymptotic conditions for T at infinity, which we do not introduce here, because we want ultimately to go to a planar approximation where they have no longer effect. Several geodesists and mathematicians have been working on this problem with different tools. We mention here only the early works by H. Moritz, T. Krarup and more recently by L. Svenson, L. Hörmander, P. Holota, F. Sansò, F. Sacerdote, G. Venuti.

The last step of this section is that we want to go to a planar approximation of (11), namely one in which the curvature radius of the ellipsoid is sent to infinity, as explained by Helmut Mortiz in [16].

In this case we easily see that $\frac{\frac{\partial \gamma}{\partial h}}{\gamma} \to 0$ so that (11) is substantially simplified.

Let us introduce then a new notation adapted to the planar approximation.

Namely we set

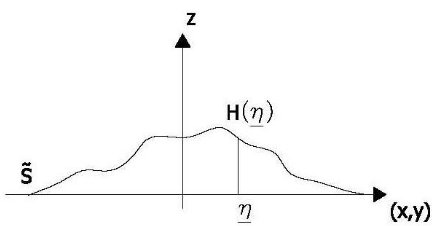

Figure 3: The planar approximation of a portion of Fig. 1

$$\xi = \begin{vmatrix} x \\ y \end{vmatrix}, \qquad \widetilde{S} \equiv \{z = H(\boldsymbol{\xi})\}$$

$$\widetilde{\Omega} \equiv \{z \geq H(\boldsymbol{\xi})\}$$

$$\begin{aligned} &T = T(\boldsymbol{\xi}, z) \\ &\nabla_0 T = \mathbf{e}_x \frac{\partial T}{\partial x} + \mathbf{e}_y \frac{\partial T}{\partial y} \equiv \nabla_\xi T \\ &T' = \frac{\partial T}{\partial z} \ . \end{aligned}$$

With this notation we have the new problem

$$\begin{cases} \Delta T = 0 & \text{in } \widetilde{\Omega} \\ T'|_{\widetilde{S}} = -\Delta g(\boldsymbol{\xi}) & \text{on } \widetilde{S} \end{cases} \tag{12}$$

where we impose on T just to be regular at infinity, namely

$$T \to 0 \text{ for } z \to \infty \ . \tag{13}$$

2 A recall of the BEM for the Neumann problem

The third Green identity for a function T harmonic in Ω, when taking the limit for $z \to H(\boldsymbol{\xi})$, reads

$$T(\boldsymbol{\xi}, H(\boldsymbol{\xi})) = \frac{1}{2\pi} \int \{T(\boldsymbol{\eta}, H(\boldsymbol{\eta})) \frac{\partial}{\partial n_\eta} \left(\frac{1}{\ell_{\xi\eta}} \right) - \frac{\partial T}{\partial n_\eta} (\boldsymbol{\eta}, H(\boldsymbol{\eta})) \frac{1}{\ell_{\xi\eta}} \} dSy \tag{14}$$

In this equation $\boldsymbol{\xi}, \boldsymbol{\eta}$ are points on the $z = 0$ plane,

$$\ell_{\boldsymbol{\xi\eta}} = [|\boldsymbol{\xi} - \boldsymbol{\eta}|^2 + (H(\boldsymbol{\xi}) - H(\boldsymbol{\eta}))^2]^{1/2} \tag{15}$$

and

$$\begin{aligned} \frac{\partial T}{\partial n_\eta}(\eta, H(\eta)) &\equiv \mathbf{n}_\eta \cdot \nabla T \Big|_{S \equiv \{z = H(\eta)\}} \equiv \\ &\equiv \mathbf{n}_\eta \cdot \nabla_0 T(\eta, H(\eta)) + \mathbf{n}_\eta \cdot \mathbf{e}_z T'(\eta, H(\eta)) \ ; \end{aligned} \tag{16}$$

similarly

$$\frac{\partial}{\partial n_\eta} \left(\frac{1}{\ell} \right) = \frac{\mathbf{n}_\eta \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) + \mathbf{n}_\eta \cdot \mathbf{e}_z (H(\boldsymbol{\xi}) - H(\boldsymbol{\eta}))}{\ell^3_{\boldsymbol{\xi\eta}}} \ . \tag{17}$$

Remark 1. It is important to realize that the Neumann operator $N_{\xi,z}$ defined as

$$N_{\mathbf{x}}\{T(\boldsymbol{\xi}, z\} \equiv \mathbf{n}(\boldsymbol{\xi}) \cdot \nabla T(\boldsymbol{\xi}, z) \tag{18}$$

acts on the harmonic potential T in a way that can be defined for any point $\mathbf{x} \equiv (\boldsymbol{\xi}, z) \in \Omega$, given the particular geometry of S. So if we further define the trace operator on S

$$\Gamma_x T(\boldsymbol{\xi}, z) \equiv T(\boldsymbol{\xi}, H(\boldsymbol{\xi})) \; , \tag{19}$$

when we write (16) we mean

$$\frac{\partial T}{\partial n_\eta}(\boldsymbol{\eta}, H(\boldsymbol{\eta})) \equiv \Gamma_x\{N_{\mathbf{x}}(T)\} \; ; \tag{20}$$

a similar consideration holds for (17), so that

$$\frac{\partial}{\partial n_y}\left(\frac{1}{\ell}\right) \equiv \Gamma_x\{\Gamma_y[N_{\mathbf{x}} \ell^{-1}_{\mathbf{x},\mathbf{y}}]\}$$

where $\mathbf{x} = (\boldsymbol{\xi}, z)$ and $\mathbf{y} = (\boldsymbol{\eta}, z')$.

Keeping the above Remark in mind, we can rewrite (14) as

$$\Gamma_{\mathbf{x}}\{T(\boldsymbol{\xi}, z)\} = \frac{1}{2\pi}\int\{\Gamma_{\mathbf{y}}(T(\boldsymbol{\eta}, z')) \cdot \Gamma_{\mathbf{x}}[\Gamma_{\mathbf{y}}(N_{\mathbf{y}}\frac{1}{\ell_{\mathbf{xy}}})] +$$
$$-\Gamma_{\mathbf{y}}[N_{\mathbf{y}}(T(\boldsymbol{\eta}, z'))]\Gamma_{\mathbf{x}}[\Gamma_{\mathbf{y}}\frac{1}{\ell_{\mathbf{xy}}}]\}dSy \tag{21}$$

Now if we assume that the normal derivative of T is given on S, i.e. that

$$g(\boldsymbol{\eta}) = -\Gamma_{\mathbf{y}}[N_{\mathbf{y}}(T(\boldsymbol{\eta}, z'))] \tag{22}$$

is a known function and we want to find T on S, namely

$$f(\boldsymbol{\xi}) = \Gamma_{\mathbf{x}}[T(\boldsymbol{\xi}, z)] \; , \tag{23}$$

we can simply rewrite (21) in the form

$$f(\boldsymbol{\xi}) - \frac{1}{2\pi}\int f(\boldsymbol{\eta})\frac{\mathbf{n}_\eta \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) + \mathbf{n}_\eta \cdot \mathbf{e}_{z'}(H(\boldsymbol{\xi}) - H(\boldsymbol{\eta}))}{\ell^3_{\boldsymbol{\xi\eta}}}dS_\eta$$
$$= \frac{1}{2\pi}\int g(\boldsymbol{\eta})\frac{1}{\ell_{\boldsymbol{\xi\eta}}}dS_\eta \; . \tag{24}$$

The right hand side of (24) is known, so this is just an integral equation of the second kind in the unknown $f(\boldsymbol{\xi})$. As we can see the kernel of this equation is singular, but not too much as we shall show in the next section, if we put some restrictive hypotheses on the regularity of S. Therefore (24) is a weakly singular equation and standard numerical techniques can be applied to its solution, for instance a spline discretization, what is typical of the BEM.

3 The role of the regularity of S

The question of the solution of singular integral equation is an old item in Mathematics dating back to the 30ies of the past century [24, 2]. Excellent classical reviews are [13, 14].

The item has got quite a renaissance in more recent times in the study of strongly elliptic systems, by using the full apparatus of analysis in Sobolev spaces that was mostly developed in the middle of 20th century. On the recent set up of the problem one can consult [12]. Here we limit ourselves to use the results of [13].

So in this section we introduce some elements of the geometric analysis of S and study the regularity of the kernel $K(\boldsymbol{\xi}, \boldsymbol{\eta})$ of equation (24), namely

$$K(\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{\mathbf{n}_\eta \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) + \mathbf{n}_\eta \cdot \mathbf{e}_{z'} (H(\boldsymbol{\xi}) - H(\boldsymbol{\eta}))}{\ell^3_{\boldsymbol{\xi}\boldsymbol{\eta}}} \tag{25}$$

We shall see then in next section how things are changed when we go over from the solution of a Neumann problem described by (24) to the solution of an oblique derivative problem.

So let us first of all note that

$$\mathbf{v}(\boldsymbol{\xi}) = \nabla_0 H(\boldsymbol{\xi}) \tag{26}$$

is a vector in the horizontal plance indicating the direction of maximum increase of the function $H(\boldsymbol{\xi})$ and its modulus $v(\boldsymbol{\xi})$ is just the tangent of the maximum inclination $I(\boldsymbol{\xi})$ of S at $\boldsymbol{\xi}$, as shown in Fig. 4 and by the formula

$$|\mathbf{v}(\boldsymbol{\xi})| = tgI(\boldsymbol{\xi}) = \sup \lim_{\eta \to \xi} \frac{|H(\boldsymbol{\eta}) - H(\boldsymbol{\xi})|}{|\boldsymbol{\eta} - \boldsymbol{\xi}|} = \lim_{\rho \to 0} \frac{|H(\boldsymbol{\xi} + \rho \mathbf{e}) - H(\boldsymbol{\xi})|}{\rho} \ . \tag{27}$$

Note that in this way $I(\boldsymbol{\xi})$ is always positive; we shall also stipulate that

$$I(\boldsymbol{\xi}) \le I_+ < \pi/2 \tag{28}$$

so that

$$|\mathbf{v}(\boldsymbol{\xi})| \le v_+ = tgI_+ < +\infty \tag{29}$$

Let us note too that the relation between the area element dS on S and its projection dS_0 on the (x, y) plane is just

$$dS_0 = \cos I dS_\eta \ . \tag{30}$$

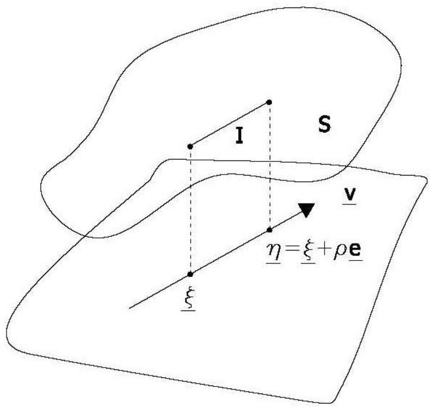

Figure 4: The inclination of S. Here $\mathbf{e}$ is a unit vector in the same direction as $\mathbf{v}$

In particular we have

$$\cos I = (1 + v(\boldsymbol{\xi})^2)^{-1/2}$$

so that for the Jacobian J we have

$$J = \frac{dS_\eta}{dS} = \frac{1}{\cos I} \leq \frac{1}{\cos I_+} = (1 + v_+^2)^{1/2} = J_+ \ . \tag{31}$$

Now, although this condition could be further generalized, we shall agree that $H(\boldsymbol{\xi})$ has bounded second order derivatives, implying also that the curvature of S has to be bounded.

So if we put

$$C(\boldsymbol{\xi}) = [\partial_{ik} H(\boldsymbol{\xi})] \ , \quad (i, k = 1, 2) \tag{32}$$

and

$$C_+ = \sup_{\boldsymbol{\xi}} \parallel C(\boldsymbol{\xi}) \parallel \ , \tag{33}$$

we have the Taylor expansion

$$\begin{aligned} H(\boldsymbol{\eta}) - H(\boldsymbol{\xi}) &= H(\boldsymbol{\xi} + \rho\mathbf{e}) - H(\boldsymbol{\xi}) = \\ &= \rho\mathbf{v}(\boldsymbol{\xi}) \cdot \mathbf{e} + \frac{1}{2}\rho^2 \mathbf{e}^T C(\boldsymbol{\xi} + \sigma\mathbf{e})\mathbf{e} \end{aligned} \tag{34}$$

with $0 < \sigma < \rho$.

Note that in this case $\mathbf{e}$ is an arbitrary direction in the (x, y) plane and $\rho\mathbf{e} = \boldsymbol{\eta} - \boldsymbol{\xi}$, so that (34) can be written

$$H(\boldsymbol{\eta}) - H(\boldsymbol{\xi}) = \mathbf{v}(\boldsymbol{\xi}) \cdot (\boldsymbol{\eta} - \boldsymbol{\xi}) + \mathcal{R}(\boldsymbol{\xi}, \boldsymbol{\eta}) \tag{35}$$

with

$$|R(\boldsymbol{\xi}, \boldsymbol{\eta})| \le \frac{1}{2} C_+ |\boldsymbol{\eta} - \boldsymbol{\xi}|^2 \ , \tag{36}$$

Now we can notice that the infinitesimal vector

$$d\mathbf{t} = d\boldsymbol{\xi} + dH\mathbf{e}_z = d\boldsymbol{\xi} + \mathbf{v} \cdot d\boldsymbol{\xi}\mathbf{e}_z$$

is tangent to S at $\boldsymbol{\xi}$ for every $d\boldsymbol{\xi}$. Therefore the vector $-\mathbf{v} + \mathbf{e}_z$ is orthogonal to all $d\mathbf{t}$, since

$$(-\mathbf{v} + \mathbf{e}_z) \cdot (d\boldsymbol{\xi} + \mathbf{v} \cdot d\boldsymbol{\xi}\mathbf{e}_z) \equiv 0 \ .$$

Accordingly, the normal to S at $\boldsymbol{\xi}$ can be written

$$\mathbf{n}_\xi = \frac{-\mathbf{v}(\boldsymbol{\xi}) + \mathbf{e}_z}{[|\mathbf{v}|^2 + 1]^{1/2}} = \cos I(-\mathbf{v} + \mathbf{e}_z) \ . \tag{37}$$

One easily realizes that this is the normal pointing in Ω. Therefore, recalling (25) and (37),

$$K(\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{\cos I(\eta)}{\ell^3_{\boldsymbol{\xi\eta}}}[-\mathbf{v}(\boldsymbol{\eta}) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) + H(\boldsymbol{\xi}) - H(\boldsymbol{\eta})]$$

so that, using (35) and (36),

$$\begin{aligned} |K(\boldsymbol{\xi}, \boldsymbol{\eta})| &\le \frac{\cos I(\boldsymbol{\eta})}{\ell^3_{\xi\eta}} \frac{1}{2} C_+ \cdot |\boldsymbol{\xi} - \boldsymbol{\eta}|^2 \le \\ &\le \frac{\frac{1}{2} C_+ \cos I(\boldsymbol{\eta})}{\ell_{\boldsymbol{\xi\eta}}} \ ; \end{aligned} \tag{38}$$

this is because

$$\frac{|\boldsymbol{\xi}-\boldsymbol{\eta}|}{\ell_{\boldsymbol{\xi}\boldsymbol{\eta}}}=\frac{|\boldsymbol{\xi}-\boldsymbol{\eta}|}{[|\boldsymbol{\xi}-\boldsymbol{\eta}|^2+(H(\boldsymbol{\xi})-H(\boldsymbol{\eta}))^2]^{1/2}}\leq 1 \ .$$

Since the kernel K is majorized by the integral kernel at the right hand side of (38) which is integrable, we can claim that under the above regularity conditions the integral equation (24) is only weakly singular and it enjoys the Fredholm alternative. That the exterior Neumann problem has a unique solution is well known and so the BEM equation (24) is well suited to approximate numerical solutions.

4 The BEM equation for Molodensky's problem

We finally go back to the equation (24), keeping the notation (23) for the unknown T on S, but going back to the definition (22) in the right hand side.

So this writes, recalling also (12) and (30),

$$\begin{aligned}
&-\frac{1}{2\pi}\int \Gamma_{\mathbf{y}}[N_{\mathbf{y}}(T)]\frac{1}{\ell_{\boldsymbol{\xi}\boldsymbol{\eta}}}dS_\eta = \qquad (39)\\
&-\frac{1}{2\pi}\int \frac{\cos I(\eta)}{\ell_{\boldsymbol{\xi}\boldsymbol{\eta}}}[-\mathbf{v}(\boldsymbol{\eta})\cdot\nabla_0 T(\boldsymbol{\eta},H(z))+T'(\boldsymbol{\eta},H(\boldsymbol{\eta}))]dS_\eta\\
&=\frac{1}{2\pi}\int \frac{1}{\ell_{\boldsymbol{\xi}\boldsymbol{\eta}}}\mathbf{v}(\boldsymbol{\eta})\cdot\nabla_0 T(\boldsymbol{\eta},H(\boldsymbol{\eta}))dS_0+\frac{1}{2\pi}\int\frac{1}{\ell_{\boldsymbol{\xi},\boldsymbol{\eta}}}\Delta g(\boldsymbol{\eta})dS_0
\end{aligned}$$

The next step is fundamental to find the correct integral equation.

Recall that by definition

$$f(\boldsymbol{\eta})=T(\boldsymbol{\eta},H(\boldsymbol{\eta}))=\Gamma_{\mathbf{y}}T(\boldsymbol{\eta},z')\ ; \qquad (40)$$

if we take the ∇_η of this relation we get

$$\begin{aligned}
\nabla_\eta\Gamma_{\mathbf{y}}T(\eta,z')=\nabla_\eta f(\eta) &= \nabla_0 T(\boldsymbol{\eta},H(\boldsymbol{\eta}))+T'(\boldsymbol{\eta},H(\boldsymbol{\eta}))\nabla_0 H(\boldsymbol{\eta}) \qquad (41)\\
&= \Gamma_{\mathbf{y}}\nabla_0 T(\boldsymbol{\eta},z')+\Gamma_{\mathbf{y}}T'(\eta,H')\nabla_0 H(\boldsymbol{\eta})
\end{aligned}$$

so that the term

$$\mathbf{v}(\boldsymbol{\eta})\cdot\nabla_0 T(\eta,H(\eta))=\mathbf{v}(\boldsymbol{\eta})\cdot\Gamma_y\nabla_0 T(\boldsymbol{\eta},z')$$

appearing in (39) can be written as

$$\begin{aligned}\mathbf{v}\cdot\nabla_0 T(\boldsymbol{\eta},H(\eta)) &= \mathbf{v}(\boldsymbol{\eta})\cdot\nabla_\eta f(\boldsymbol{\eta})-|\mathbf{v}(\boldsymbol{\eta})|^2 T'(\boldsymbol{\eta},H(\boldsymbol{\eta}))=\\ &= \mathbf{v}(\boldsymbol{\eta})\cdot\nabla_\eta f(\boldsymbol{\eta})+|\mathbf{v}(\boldsymbol{\eta})|^2\Delta g(\eta)\ .\end{aligned} \tag{42}$$

Substituting and integrating by parts on the horizontal plane, we get

$$-\frac{1}{2\pi}\int \Gamma_y[N_{\mathbf{y}}(T)]\frac{1}{\ell_{\boldsymbol{\xi\eta}}}dS_y = \tag{43}$$

$$=\frac{1}{2\pi}\int\frac{1}{\ell_{\boldsymbol{\xi\eta}}}\mathbf{v}(\boldsymbol{\eta})\cdot\nabla_\eta f(\boldsymbol{\eta})dS_0+\frac{1}{2\pi}\int\frac{1+|\mathbf{v}(\boldsymbol{\eta})|^2}{\ell_{\boldsymbol{\xi\eta}}}\Delta g(\boldsymbol{\eta})dS_0$$

$$=-\frac{1}{2\pi}\int(\nabla_\eta\cdot\left(\frac{\mathbf{v}(\boldsymbol{\eta})}{\ell_{\boldsymbol{\xi\eta}}}\right)f(\boldsymbol{\eta})dS_0+h(\boldsymbol{\xi}) \tag{44}$$

where $h(\boldsymbol{\xi})$ is a known function, i.e.

$$h(\boldsymbol{\xi})=\frac{1}{2\pi}\int\frac{1+v^2(\boldsymbol{\eta})}{\ell_{\boldsymbol{\xi\eta}}}\Delta g(\boldsymbol{\eta})dS_0\ . \tag{45}$$

Furthermore one has

$$\nabla_\eta\cdot\frac{\mathbf{v}(\boldsymbol{\eta})}{\ell_{\boldsymbol{\xi\eta}}}=\frac{\nabla_\eta\cdot\mathbf{v}(\eta)}{\ell_{\boldsymbol{\xi\eta}}}+\mathbf{v}(\boldsymbol{\eta})\cdot\nabla_\eta\frac{1}{\ell_{\xi\eta}}= \tag{46}$$

$$=\frac{\Delta_\eta H(\eta)}{\ell_{\boldsymbol{\xi\eta}}}+\frac{\mathbf{v}(\boldsymbol{\eta})\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})+(H(\boldsymbol{\xi})-H(\boldsymbol{\eta}))(\mathbf{v}(\boldsymbol{\eta}))^2}{\ell^3_{\boldsymbol{\xi\eta}}}$$

For the coming discussion, it is convenient to use the notation

$$|\boldsymbol{\xi}-\boldsymbol{\eta}|=\rho\ ,$$

as already done in §3, and

$$\frac{\boldsymbol{\xi}-\boldsymbol{\eta}}{r}=\boldsymbol{\vartheta}$$

as in the classical text of Mikhlin.

Recalling (35), it is immediate to prove that

$$1\leq\frac{\ell_{\boldsymbol{\xi\eta}}}{\rho}\leq C\quad<+\infty$$

so that

$$\ell_{\boldsymbol{\xi\eta}} = 0(\rho) \ . \tag{47}$$

Therefore an account of the hypotheses of boundedness of the second derivatives of $H(\boldsymbol{\xi})$ (cf. (32), (33)) one has

$$\frac{\Delta_\eta H(\boldsymbol{\eta})}{\ell_{\boldsymbol{\xi\eta}}} = O(\rho^{-1}) \ , \tag{48}$$

showing that this kernel is only weakly singular and then the corresponding integral operator is compact, for instance, in L^2.

We study now the second part of the kernel (46), namely

$$\mathcal{H}(\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{\mathbf{v}(\boldsymbol{\eta}) \cdot (\boldsymbol{\xi} - \boldsymbol{\eta}) + (H(\boldsymbol{\xi}) - H(\boldsymbol{\eta}))v^2(\boldsymbol{\eta})}{[\rho^2 + (H(\boldsymbol{\xi}) - H(\eta))^2]^{3/2}} \ . \tag{49}$$

Our purpose is to prove that (49) can be decomposed into two kernels, one of which weakly singular and the other strongly singular, but such that the theory of [13] can be replied.

Proposition 1. We have

$$\mathcal{H}(\boldsymbol{\xi}, \boldsymbol{\eta}) = W(\boldsymbol{\xi}, \boldsymbol{\eta}) + S(\boldsymbol{\xi}, \boldsymbol{\eta}) \tag{50}$$

where

$$|W(\boldsymbol{\xi}, \boldsymbol{\eta})| = O(\rho^{-1}) \ ,$$

i.e. W is weakly singular, while

$$S(\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{f(\boldsymbol{\xi}, \boldsymbol{\vartheta})}{\rho^2} \ , \tag{51}$$

where $f(\boldsymbol{\xi}, \boldsymbol{\vartheta})$, the so-called *characteristic of the kernel*, is bounded, continuous in $\boldsymbol{\vartheta}$ for each fixed $\boldsymbol{\xi}$ and such that the fundamental condition holds

$$\int_0^{2\pi} f(\boldsymbol{\xi}, \boldsymbol{\vartheta}) d\vartheta = 0 \ , \tag{52}$$

where ϑ is just the angular anomaly of the vector $\boldsymbol{\vartheta}$, running along the unit circle.

Proof. We first note that, also recalling (35),

$$W_0(\xi,\eta) = \mathcal{H}(\boldsymbol{\xi},\boldsymbol{\eta}) + \mathcal{H}(\boldsymbol{\eta},\boldsymbol{\xi}) = \tag{53}$$
$$= \frac{[\mathbf{v}(\boldsymbol{\eta}) - \mathbf{v}(\boldsymbol{\xi})]\cdot(\boldsymbol{\xi}-\boldsymbol{\eta}) + [\mathbf{v}(\boldsymbol{\xi})\cdot(\boldsymbol{\xi}-\boldsymbol{\eta}) + \mathcal{R}(\boldsymbol{\xi},\boldsymbol{\eta})][v^2(\boldsymbol{\eta}) - v^2(\boldsymbol{\xi})]}{\ell^3_{\boldsymbol{\xi}\boldsymbol{\eta}}} .$$

Since

$$|\mathbf{v}(\boldsymbol{\eta}) - \mathbf{v}(\boldsymbol{\xi})| = O(\rho)$$

and

$$|v^2(\boldsymbol{\eta}) - v^2(\boldsymbol{\xi})| \le |\mathbf{v}(\boldsymbol{\eta}) - \mathbf{v}(\boldsymbol{\xi})||\mathbf{v}(\boldsymbol{\eta}) + \mathbf{v}(\boldsymbol{\xi})| = O(\rho) ,$$

we see that the numeration in (53) is $O(\rho^2)$ and so

$$|W_0(\boldsymbol{\xi},\boldsymbol{\eta})| = O(\rho^{-1}) , \tag{54}$$

i.e. it is a weakly singular kernel. So we can reduce ourselves to study

$$-\mathcal{H}(\boldsymbol{\eta},\boldsymbol{\xi}) = \frac{\mathbf{v}(\boldsymbol{\xi})\cdot(\boldsymbol{\xi}-\boldsymbol{\eta}) + [\mathbf{v}(\boldsymbol{\xi})\cdot(\boldsymbol{\xi}-\boldsymbol{\eta}) + \mathcal{R}(\boldsymbol{\xi},\boldsymbol{\eta})]v^2(\boldsymbol{\xi})}{\ell^3_{\boldsymbol{\xi}\boldsymbol{\eta}}} \tag{55}$$

Since $\mathcal{R}(\boldsymbol{\xi},\boldsymbol{\eta}) = O(\rho^2)$ we see that

$$W_1(\boldsymbol{\eta},\boldsymbol{\xi}) = \frac{\mathcal{R}(\boldsymbol{\xi},\boldsymbol{\eta})\cdot v^2(\boldsymbol{\xi})}{\ell^3_{\boldsymbol{\xi}\boldsymbol{\eta}}} = O(\rho^{-1}) , \tag{56}$$

namely it is weakly singular.

So we have only to study

$$\begin{aligned} \mathcal{L}(\boldsymbol{\eta},\boldsymbol{\xi}) &= \frac{\mathbf{v}(\boldsymbol{\xi})\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})(1+v^2(\boldsymbol{\xi}))}{\rho^3\left[1+\left(\frac{\mathbf{v}(\boldsymbol{\xi})\cdot(\boldsymbol{\xi}-\boldsymbol{\eta})+\mathcal{R}(\boldsymbol{\xi},\boldsymbol{\eta})}{\rho}\right)\right]^{3/2}} = \\ &= \frac{1}{\rho^2}\frac{\mathbf{v}(\boldsymbol{\xi})\cdot\boldsymbol{\vartheta}(1+v^2(\boldsymbol{\xi}))}{\left[1+\left(\mathbf{v}(\boldsymbol{\xi})\cdot\boldsymbol{\vartheta}+\frac{\mathcal{R}}{\rho}\right)^2\right]^{3/2}} \end{aligned} \tag{57}$$

so we finally write

$$\begin{aligned} \mathcal{L}(\boldsymbol{\eta},\boldsymbol{\xi}) &= \frac{\mathbf{v}(\boldsymbol{\xi})\cdot\boldsymbol{\vartheta}(1+v^2(\boldsymbol{\xi}))}{\rho^2}\left[\frac{1}{\left[1+(\mathbf{v}(\boldsymbol{\xi})\cdot\boldsymbol{\vartheta}+\frac{\mathcal{R}}{\rho})^2\right]^{1/2}}+\right. \\ &\left.-\frac{1}{[1+(\mathbf{v}(\boldsymbol{\xi})\cdot\boldsymbol{\vartheta})^2]^{3/2}}\right] + \frac{\mathbf{v}(\boldsymbol{\xi})\cdot\boldsymbol{\vartheta}(1+v^2(\boldsymbol{\xi}))}{\rho^2[1+(\mathbf{v}(\boldsymbol{\xi})\cdot\boldsymbol{\vartheta})^2]^{3/2}} \\ &\equiv W_2(\boldsymbol{\eta},\boldsymbol{\xi}) + S(\boldsymbol{\xi},\boldsymbol{\eta}) . \end{aligned} \tag{58}$$

We can further notice that, since

$$\frac{\mathcal{R}}{\rho} = O(\rho)$$

we have too

$$W_2(\boldsymbol{\eta}, \boldsymbol{\xi}) = O(\rho^{-1}) \ , \tag{59}$$

i.e. this is a weakly singular kernel. On the contrary

$$S(\boldsymbol{\xi}, \boldsymbol{\eta}) = \frac{1}{\rho^2} \frac{\mathbf{v}(\boldsymbol{\xi}) \cdot \boldsymbol{\vartheta}(1 + v^2(\boldsymbol{\xi}))}{[1 + (\mathbf{v}(\boldsymbol{\xi}) \cdot \vartheta)^2]^{3/2}} \tag{60}$$

is in fact a strongly singular kernel. This however has the canonical form (51) and the characteristic is both bounded and continuous in ϑ. Therefore (50) and (51) are proved with

$$W(\boldsymbol{\xi}, \boldsymbol{\eta}) = W_0(\boldsymbol{\xi}, \boldsymbol{\eta}) + W_1(\boldsymbol{\eta}, \boldsymbol{\xi}) + W_2(\boldsymbol{\eta}, \boldsymbol{\xi}) \ .$$

Moreover since

$$f(\boldsymbol{\xi}, \boldsymbol{\vartheta}) = \frac{\mathbf{v}(\boldsymbol{\xi}) \cdot \boldsymbol{\vartheta}(1 + v^2(\boldsymbol{\xi}))}{[1 + (\mathbf{v}(\boldsymbol{\xi}) \cdot \boldsymbol{\vartheta})^2]^{3/2}}$$

is an odd function in $\boldsymbol{\vartheta}$, the condition (52) is indeed satisfied. □

Due to the above proposition, we know from a theorem by Tricomi ([13], Theorem 1.5) that the integral operator with strongly singular kernel (60) is a bounded operator in L^2.

The next property we need is that the operator $I - A$, with A the singular integral operator

$$AT = \frac{1}{2\pi} \int \frac{f(\boldsymbol{\xi}, \boldsymbol{\vartheta})}{\rho^2} T(\boldsymbol{\eta}) d_2\eta \ , \tag{61}$$

has index 0, i.e. it enjoys the Fredholm alternative property. This can be based on the symbol of $I - A$, defined as (see [13], §22, (1))

$$\begin{aligned} \Psi(\boldsymbol{\xi}, \vartheta) = & \ \text{Simb } (I - A) = 1 - \Phi(\boldsymbol{\xi}, \vartheta) = \\ = 1 - & \frac{1}{2\pi} \int \{\log \frac{1}{|\cos(\vartheta - \vartheta_0)|} - i\frac{\pi}{2} \text{ sign } \cos(\vartheta - \vartheta_0)\} f(\boldsymbol{\xi}, \vartheta_0) d\vartheta_0 \ . \end{aligned} \tag{62}$$

In this formula the vector $\boldsymbol{\vartheta}$ is identified with the corresponding angle ϑ and so it is for $\boldsymbol{\vartheta}_0$ and ϑ_0.

According to Theorem 1.37 of [13] $\Psi(\boldsymbol{\xi}, \vartheta)$ has to be a smooth function of ϑ (e.g. to have square integrable third derivatives) and to be such that

$$\inf |\Psi(\boldsymbol{\xi}, \vartheta)| > 0 \ . \tag{63}$$

As for the first condition it gives no harm, since we can write

$$\Phi(\boldsymbol{\xi}, \vartheta) = \frac{1}{2\pi} \int \{\log \frac{1}{|\cos \vartheta'|} - i\frac{\pi}{2} \text{ sign } \cos \vartheta\} f(\boldsymbol{\xi}, \vartheta - \vartheta') d\vartheta' \tag{64}$$

and

$$f(\boldsymbol{\xi}, \vartheta - \vartheta') = \frac{v(\boldsymbol{\xi}) \cos(\vartheta - \vartheta')}{]1 + v^2(\boldsymbol{\xi}) \cos^2(\vartheta - \vartheta')]^{3/2}} \tag{65}$$

is an analytic function of ϑ.

Moreover the relation (63) holds for the characteristic (65). Let us note that

$$\begin{aligned} &|\Psi(\boldsymbol{\xi}, \vartheta) \geq |Re\Psi(\boldsymbol{\xi}, \vartheta)| \geq \\ &\geq 1 - \frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{|\cos \vartheta'|} |f(\boldsymbol{\xi}, \vartheta - \vartheta')| d\vartheta' = 1 - I \ . \end{aligned} \tag{66}$$

On the other hand it is elementary to verify that in $x \geq 0$

$$\frac{x}{[1 + x^2]^{3/2}} \leq \frac{2}{3\sqrt{3}} \ ,$$

so that, exploiting a standard integral,

$$\begin{aligned} I &\leq \frac{2}{3\sqrt{3}} \frac{4}{2\pi} \int_0^{\pi/2} \log \frac{1}{\cos \vartheta'} d\vartheta' = \\ &= \frac{2}{3\sqrt{3}} \frac{2}{\pi} \cdot \frac{\pi}{2} \log 2 = \frac{2 \log 2}{3\sqrt{3}} = 1 \\ &= 0{,}26780, \ 2678 < 1 \end{aligned}$$

this shows that (63) is satisfied.

Summarizing the findings of this section, we can say that the problem of Molodensky written in planar approximation can be cast into the form of a strongly singular integral equation, namely, by using formulas (46), (45), (44), (39), (37) into the third Green identity (21),

$$f(\boldsymbol{\xi}) = \frac{1}{2\pi}\int \frac{-2\mathbf{v}(\boldsymbol{\eta})\cdot(\boldsymbol{\xi}-\boldsymbol{\eta}) + (H(\boldsymbol{\xi}) - H(\boldsymbol{\eta})(1-v^2(\boldsymbol{\eta}))}{\ell^3_{\boldsymbol{\xi}\boldsymbol{\eta}}} f(\eta)dS_0$$
$$-\frac{1}{2\pi}\int \frac{\Delta_0 H(\boldsymbol{\eta})}{\ell_{\boldsymbol{\xi}\boldsymbol{\eta}}} f(\boldsymbol{\eta})dS_0 + h(\boldsymbol{\xi}) \; . \tag{67}$$

The known term $h(\boldsymbol{\xi})$, given by (45), is certainly L^2 if so is Δg; more precisely $h(\boldsymbol{\xi})$ is in $H^{1,2}$ when Δg is in L^2. Since the alternative holds for (67) in L^2, thanks to Proposition 1, if we prove that the original problem (12) has only one solution when $\Delta g \in L^2$, as we do in the Appendix, we can say that (67) has one and only one solution. Even more, since we prove in the Appendix that when $\Delta g \in L^2$, then $T \in H^{1,2}$ we arrive at the following Theorem:

Theorem 1. *The problem of Molodensky in planar approximation (12) can be equivalently put in the form of a strongly singular integral equation (67). Such an equation has one and only one solution $T \in H^{1,2}$ for every $\Delta g \in L^2$.*

5 Conclusions

The linearized problem of Molodensky in planar approximation can be transformed into a unique integral equation connecting Δg, known on the telluroid, with the anomalous potential T on the same surface and the corresponding height anomaly, allowing the reconstruction of the unknown *surface of the Earth.*

The integral equation is necessarily strongly singular, however it is possible to show that it enjoys the Fredholm alternative property. A general theorem of uniqueness and regularity of the solution completes the analysis of the problem in a satisfactory way.

Essentially the same analysis with a few changes of the terms of (67) can be performed in spherical approximation. The mentioned stability theorem justifies a direct application of the BEM to (67), thus providing a new numerical tools for the solution of the problem.

Appendix

In this Appendix we want to prove that any solution of the problem (see (12))

$$\begin{cases} \Delta T = 0 & \text{in } \widetilde{\Omega} \equiv \{z > H(\boldsymbol{\xi})\} \\ -T'(\boldsymbol{\xi}, H(\boldsymbol{\xi})) = \Delta g & \text{on } \widetilde{S} \ , \end{cases} \tag{68}$$

regular for $h \to \infty$, when $\Delta g \in L^2(R^2)$ is unique and it is even in $H^{1,2}(R^2)$.

Proof. That the solution of (68) is unique is immediate. In fact if T is harmonic in $\widetilde{\Omega}$, then so is T'. But if $T'|_{\widetilde{S}} = 0$, then $T' \equiv 0$ in $\widetilde{\Omega}$. However this means that

$$T(\boldsymbol{\xi}, z) \equiv F(\boldsymbol{\xi}) \tag{69}$$

for some F; but we must also have

$$\lim_{z \to \infty} T(\boldsymbol{\xi}, z) = 0 \tag{70}$$

because of the regularity hypothesis, so it must be $T \equiv 0$ in $\widetilde{\Omega}$.

That $\Gamma_{\mathbf{x}}(T) \in H^{1,2}(R^2)$, namely that $\Gamma_{\mathbf{x}}(\nabla T) \in L^2(R^2)$, comes from the energy integral adaptation to the Cartesian geometry here used for the planar approximation [21] pag. 672. In fact we have the identity

$$\begin{aligned} \nabla \cdot (T' \nabla T) &= (\frac{\partial}{\partial z} \nabla T) \cdot \nabla T \\ &= \frac{1}{2} \frac{\partial}{\partial z} |\nabla T|^2 \ . \end{aligned} \tag{71}$$

By integrating (71) in $\widetilde{\Omega}$ we receive

$$\begin{aligned} -\frac{1}{2} \int dS_0 \Gamma_{\mathbf{x}} |\nabla T|^2 &= \int dS_0 \int_{H(\boldsymbol{\xi})}^{+\infty} dz \frac{1}{2} \frac{\partial}{\partial z} |\nabla R|^2 = \\ &= -\int \Gamma_{\mathbf{x}}(T') \Gamma_{\mathbf{x}}[N(T)] dS \ . \end{aligned} \tag{72}$$

Since

$$-\Gamma_{\mathbf{x}}(T') = \Delta g$$

and

$$dS = JdS_0 = \frac{1}{\cos I} dS_0$$

(72) yields

$$\int dS_0 \Gamma_{\mathbf{x}} |\nabla T|^2) = 2 \int dS_0 \Delta g J \Gamma_{\mathbf{x}} [N(T)] \tag{73}$$

Therefore, recalling the notation (31),

$$\begin{aligned} \int dS_0 \Gamma_{\mathbf{x}} |\nabla T|^2 \leq 2J_+ [\int dS_0 \Delta g^2(\boldsymbol{\xi})] \cdot \\ \cdot \int dS_0 \Gamma_{\mathbf{x}} \left[N(T)]^2 \right]^{1/2} \end{aligned} \tag{74}$$

Since obviously

$$\int dS_0 \Gamma_{\mathbf{x}} [N(T)]^2 \leq \int dS_0 \Gamma_{\mathbf{x}} |\nabla T|^2$$

by simplifying (74), we find

$$[\int dS_0 \Gamma_{\mathbf{x}} |\nabla T|^2]^{1/2} \leq 2J_+ [\int dS_0 \Delta g^2(\boldsymbol{\xi})]^{1/2} ,$$

namely

$$\Delta g \in L^2(R^2) \Rightarrow T \in H^{1,2}(R^2) .$$

as it was to be proved. □

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