System Dynamics of Polar Motion and length of Day Variation

E. W. Grafarend
Geodätisches Institut
Universität Stuttgart
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Erik W. Grafarend (MLS), Stuttgart

System-Analyse der Polbewegung und der Tageslängenänderung

Im statistischen Mittel werden die aller meisten wissenschaftlich-geodätischen Beiträge zum Thema Polbewegung und Tageslängenänderung geschrieben. Zahlreiche innere und äußere Effekte beein-flussen die Drehimpulsbilanz des Planeten Erde, namentlich Polbewegung/Tageslängenänderung. Einen ersten Überblick gibt das wöchentliche Bulletin des IERS („International Earth Rotation Service"), welches an interessierte Mitglieder der Internationalen Assoziation der Geodäsie („IAG/IUGG") geschickt wird.

Mein Vortrag beschäftigt mit einem modernen Zweig der systematischen Analyse, kurz System-Analyse (System Analysis) von PoM/LoD („polar motion", „length-of-day variation") . Auf der Basis der Liouville-Gleichung, der Störungstheorie der dynamischen Euler-Gleichung des Drehimpulses eines im Allgemeinen deformierbaren Körpers ergibt sich ein inhomogenes System von Integro-differentialgleichungen erster Ordnung. Unter der Annahme eines viskoelastischen Körpers einer homogen geschichteten Erde („homogeneous spherical shells") wird der Liouville-Operator spektral analysiert. Einem Vorschlag von M. Schneider (München) aus dem Jahre 1999 folgend wird das System von Differentialgleichungen erster Ordnung in ein System von Differentialgleichungen zweiter Ordnung überführt. Die klare Analyse ergibt die folgende Interpretation:
(i) Die Polbewegung stellt einen angeregten, gekoppelten, gedämpften elliptischen Oszillator dar,
(ii) Das Tageslängenänderung wird beschrieben als ein angeregtes, gedämpftes nicht-periodisch bewegtes System.
Lösungen werden im Laplace- und Fourier Gebiet diskutiert. Neue Lösungen betreffen dynamische "Wavelets" und "Fractals".

I E R S   B U L L E T I N - A

Rapid Service/Prediction of Earth Orientation

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GENERAL INFORMATION:

To receive this information electronically, contact:
ser7@maia.usno.navy.mil or use
<http://maia.usno.navy.mil/docrequest.html>

MJD = Julian Date - 2 400 000.5 days

UT2-UT1 = 0.022 sin(2*pi*T) - 0.012 cos(2*pi*T)
- 0.006 sin(4*pi*T) + 0.007 cos(4*pi*T)
where pi = 3.14159265... and T is the date in Besselian years.

TT = TAI + 32.184 seconds

DUT1= (UT1-UTC) transmitted with time signals
= -0.1 seconds beginning 21 Nov 2013 at 0000 UTC
= -0.2 seconds beginning 20 Feb 2014 at 0000 UTC

Beginning 1 July 2012:
TAI-UTC = 35.000 000 seconds

Please note that daily and Bulletin A EOP data can be obtained from the primary Earth Orientation (EO) servers at:
http://maia.usno.navy.mil ftp://maia.usno.navy.mil
and from the backup EO server at:
http://toshi.nofs.navy.mil ftp://toshi.nofs.navy.mil

There will NOT be a leap second introduced in UTC on 30 June 2014.

The contributed observations used in the preparation of this Bulletin are available at <http://www.usno.navy.mil/USNO/earth-orientation/eo-info/general/input-data>. The contributed analysis results are based on data from Very Long Baseline Interferometry (VLBI), Satellite Laser Ranging (SLR), the Global Positioning System (GPS) satellites, Lunar Laser Ranging (LLR), and meteorological predictions of variations in Atmospheric Angular Momentum (AAM).

COMBINED EARTH ORIENTATION PARAMETERS:

IERS Rapid Service

			MJD	x	error	y	error	UT1-UTC	error
				"	"	"	"	s	s
14	1	17	56674	0.02664	.00009	0.33374	.00009	-0.112586	0.000012
14	1	18	56675	0.02584	.00009	0.33507	.00009	-0.113490	0.000013
14	1	19	56676	0.02483	.00009	0.33633	.00009	-0.114549	0.000013
14	1	20	56677	0.02395	.00009	0.33737	.00009	-0.115715	0.000012
14	1	21	56678	0.02378	.00009	0.33822	.00009	-0.116948	0.000011
14	1	22	56679	0.02409	.00009	0.33902	.00009	-0.118220	0.000011

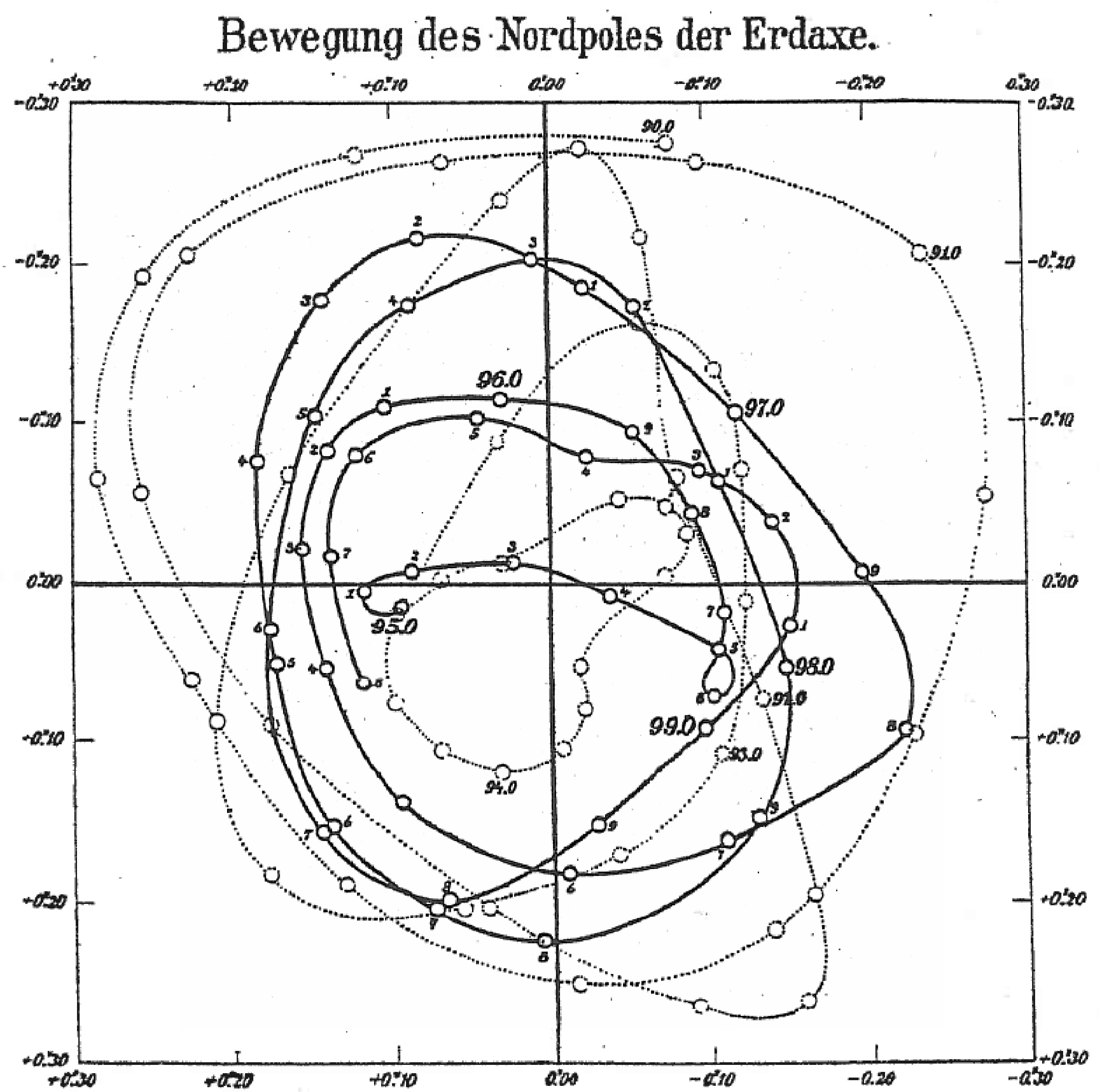


Figure:

Polar motion during the years 1890 – 1900 from F. Klein and A. Sommerfeld, The Theory of Gyroscopic Motion, published 1897-1900 , reprint of the book by 1965.

Box 1

Principle of Balance of Moment of Momentum

(Angular Momentum)

in

a quasi-body fixed frame of reference

(rotating reference frame)

first Liouville perturbation

"three constituents of angular momentums"

$$L_0 + L_1 + L_2$$

$$L_1 + L_2 = L + \delta L$$

"spin-orbit coupling"

$$D_t L_0 + \Omega \times L_0 = M_0$$

$$L_0 := M\,(\mathrm{v}_0 \times x_0)$$

orbit angular momentum

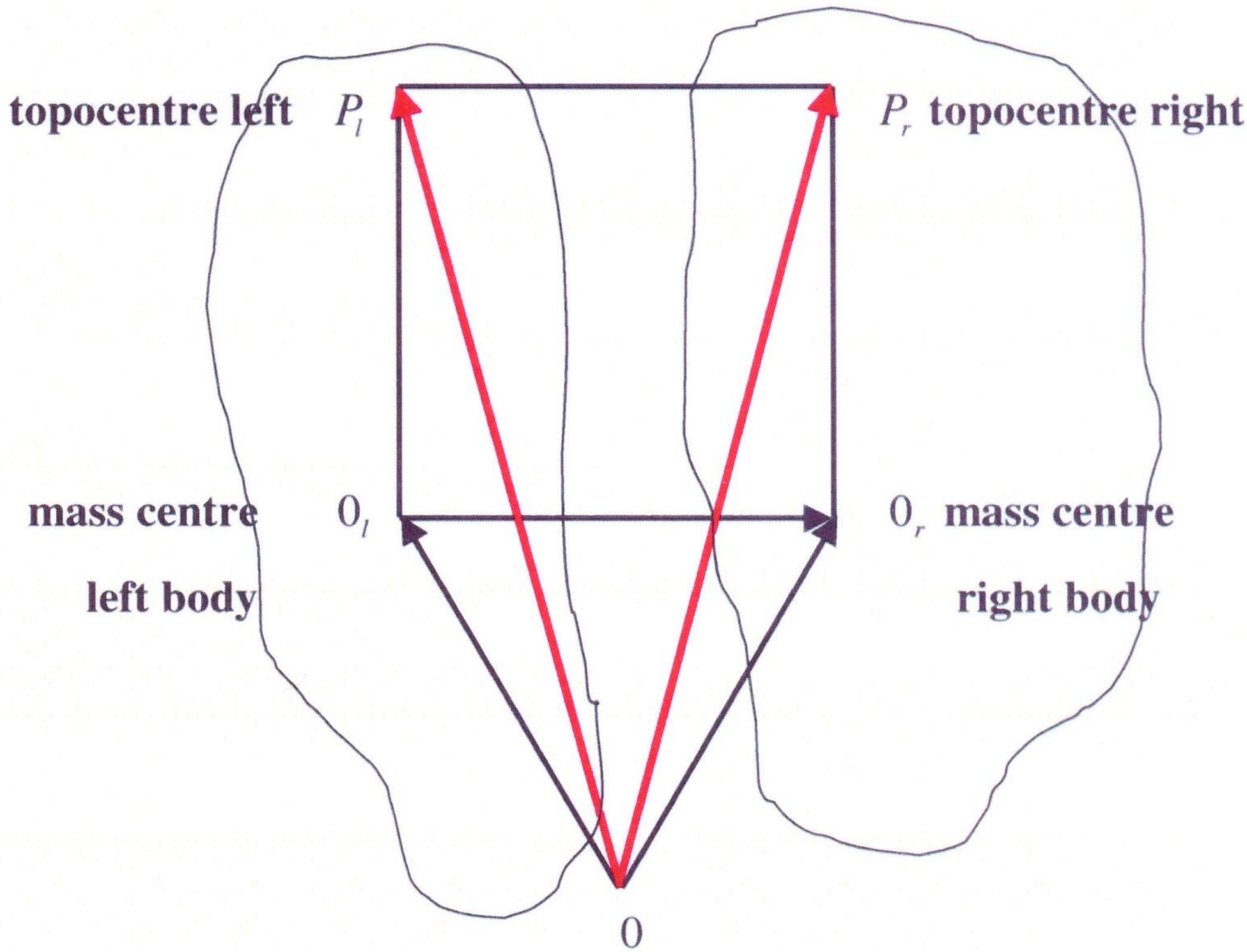
"rotation-deformation coupling"

$$D_t(L_1 + L_2) + \Omega \times (L_1 + L_2) = M_1 + M_2$$

$$J_{kl} D_t \Omega_l + (D_t J_{kl}) \Omega_l + \delta_{ijk} \Omega_i J_{jm} \Omega_m + D_t \delta L_k + \delta_{ijk} \Omega_i \delta L_j = M_k$$

Figure 1

Decomposition of the velocity field $\mathbf{v}(x,t)$, commutation diagrams, P-Diagram



$$\omega x_1 := \delta\omega_1, \quad \omega x_2 := \delta\omega_2, \quad \omega x_3 := \delta\omega_3$$

<u>1st</u>: $A\omega \dot{x}_1 + \omega^2 (C - B) x_2 + \omega\delta \dot{j}_{1,3} - \omega^2 \delta j_{2,3} + \delta \dot{L}_1 - \omega\delta L_2 = x_1$

<u>2nd</u>: $B\omega \dot{x}_2 - \omega^2 (A - C) x_1 + \omega\delta \dot{j}_{2,3} + \omega^2 \delta j_{3,1} + \delta \dot{L}_2 - \omega\delta L_1 = x_2$

<u>3rd</u>: $C\omega \dot{x}_3 + \omega\delta \dot{j}_{3,3} + \delta \dot{L}_3 = x_3$

Mc Cullagh

$$\delta j_{i,j} \longrightarrow \delta\omega_i$$

Box 6:

Love-Shida hypothesis, homogeneous spherical shell, prestressed viscoelastic Earth model in the time domain (Earth radius R)

$$\delta\omega_{2,m}(t) = \left[1 + k_2(\mathbf{load}, elastic)\right]\delta u_{2,m}(load)$$

$$+\int_0^t k_{2,R}(load, t-t')\delta u_{2,m}(\mathbf{load}, t')\, dt' + k_2(tidal, elastic)\delta u_{2,m}(tid, t)$$

$$+\int_0^t k_{2,R}(tid, t-t')\delta u_{2,m}(\mathbf{tid}, t')dt' + k_2(\mathbf{cent}, elastic)\delta u_{2,m}(cent, t)$$

$$+\int_0^t k_{2,R}(\mathbf{cent}, t-t')\delta u_{2,m}(\mathbf{cent}, t')(t')dt$$

"$k_2(elastic)$ as a *dimensionless constant*: instantaneous reaction to the action of the excitation function"

"$k_{2,R}(t-t')$ is a Love viscoelastic kernel funktion on the terrestrial sphere S^2 of dimension $1/time$. For a R homogeneous spherical shell viscoelastic Earth model the Love kernel function $k_{2,R}(t-t')$ can be represented by

$$k_{2,R} = \sum_{j=1}^{J} k_j \exp(-s_j t).$$

Numeric example:

Parameter of an Earth model with 5 layers:

radius [m]	density [kg/m^3]	shear modulus [$kg/(m\,s^2)$]	dyn. viscosity [$kg/(m\,s)$]
0000000.0			
	10932.0	$0.0000 \cdot 10^{+00}$	$0.0000 \cdot 10^{+00}$
3480000.0			
	4878.0	$0.2190 \cdot 10^{+12}$	$0.1000 \cdot 10^{+22}$
5701000.0			
	3857.0	$0.1060 \cdot 10^{+12}$	$0.1000 \cdot 10^{+22}$
5951000.0			
	3434.0	$0.7270 \cdot 10^{+11}$	$0.1000 \cdot 10^{+22}$
6250000.0			
	3184.0	$0.6020 \cdot 10^{+11}$	$0.1000 \cdot 10^{+25}$
6371000.0			

roots of the secular determinant of degree 2:

Nr.	$1/s_j$ time of relaxation [year]	s_j inverse time of relaxation [kyr^{-1}]
1	250.9310	$-0.3985 \cdot 10^{+01}$
2	282.2692	$-0.3542 \cdot 10^{+01}$
3	352.8929	$-0.2833 \cdot 10^{+01}$
4	402.8933	$-0.2482 \cdot 10^{+01}$
5	494.2672	$-0.2023 \cdot 10^{+01}$
6	2224.9892	$-0.4494 \cdot 10^{+00}$
7	9083.7657	$-0.1100 \cdot 10^{+00}$
8	530740.5736	$-0.1884 \cdot 10^{-02}$
9	708982.2371	$-0.1410 \cdot 10^{-02}$
10	28063129.4519	$-0.3563 \cdot 10^{-04}$
11	592709956.3718	$-0.1687 \cdot 10^{-05}$

Load and Love number (components of the distinct frequencies):

	load number	Love number	
	$k_{2,el}(load)$	$k_{2,el}(tide, cent)$	
	$\mathbf{-0.2462 \cdot 10^{+00}}$	$\mathbf{0.3050 \cdot 10^{+00}}$	[non-dimensional]
Nr.	$k_{2,R}(load)$	$k_{2,R}(tide, cent)$	[kyr^{-1}]
1	$\mathbf{-0.1155 \cdot 10^{+00}}$	$\mathbf{0.1807 \cdot 10^{+00}}$	
2	$\mathbf{-0.9956 \cdot 10^{-01}}$	$\mathbf{0.1136 \cdot 10^{+00}}$	
3	$\mathbf{-0.2743 \cdot 10^{+00}}$	$\mathbf{0.3671 \cdot 10^{+00}}$	
4	$\mathbf{-0.7762 \cdot 10^{-01}}$	$\mathbf{0.8120 \cdot 10^{-01}}$	
5	$\mathbf{-0.3350 \cdot 10^{+00}}$	$\mathbf{0.4265 \cdot 10^{+00}}$	
6	$\mathbf{-0.1409 \cdot 10^{+00}}$	$\mathbf{0.8122 \cdot 10^{-01}}$	
7	$\mathbf{-0.2190 \cdot 10^{-03}}$	$\mathbf{0.8207 \cdot 10^{-03}}$	
8	$\mathbf{-0.2396 \cdot 10^{-05}}$	$\mathbf{0.2261 \cdot 10^{-04}}$	
9	$\mathbf{-0.1077 \cdot 10^{-03}}$	$\mathbf{0.1524 \cdot 10^{-04}}$	
10	$\mathbf{-0.2219 \cdot 10^{-06}}$	$\mathbf{0.1124 \cdot 10^{-07}}$	
11	$\mathbf{-0.5612 \cdot 10^{-08}}$	$\mathbf{0.1123 \cdot 10^{-09}}$	

Box 10

Polar motion equations in the time domain

„system of intergro-differential equations of first order evolutionary equations"

$$\underline{\textbf{1st}}:\ \dot{x}_1 = \frac{\omega\left[B - C + \dfrac{\omega^2 R^5 k_2(cent, elastic)}{3g}\right]}{A + \dfrac{\omega^2 R^5 k_2(cent, elastic)}{3g}} x_2$$

$$- \frac{\dfrac{\omega^2 R^5 k_{2,R}(0, cent)}{3g}}{\dfrac{A + \omega^2 R^5 k_2(cent, elastic)}{3g}} x_1$$

$$- \left[A + \frac{\omega^2 R^5 k_2(cent, elastic)}{3g}\right]^{-1} \int_0^t \dot{k}_2(t - t', cent) x_1(t') dt'$$

$$+ \left[A + \frac{\omega^2 R^5 k_2(cent, elastic)}{3g}\right]^{-1} f$$

$$\textbf{2nd}: \dot{x}_2 = -\frac{\omega\left[C - A - \frac{\omega^2 R^5 k_2(cent, elastic)}{3g}\right]}{B + \frac{\omega^2 R^5 k_2(cent, elastic)}{3g}} x_1$$

$$-\frac{\frac{\omega^2 R^5 k_{2,R}(0, cent)}{3g}}{B + \frac{\omega^2 R^5 k_2(cent, elastic)}{3g}} x_2$$

$$-\left[B + \frac{\omega^2 R^5 k_2(cent, elastic)}{3g}\right]^{-1} \int_0^t \dot{k}_2(t - t', cent) x_2(t') dt'$$

$$+\left[B + \frac{\omega^2 R^5 k_2(cent, elastic)}{3g}\right]^{-1} \mathrm{g}$$

"system equations"

$$\dot{x} = Ax + f(x) + b$$

$$a_{11} = -\left[A + \frac{\omega^2 R^5 k_2(cent, elastic)}{3g}\right]^{-1} \frac{\omega^2 R^5 k_{2,R}(0, cent)}{3g}$$

$$a_{12} = \left[A + \frac{\omega^2 R^5 k_2(cent, elastic)}{3g}\right]^{-1} \omega\left[B - C + \frac{\omega^2 R^5 k_2(cent, elastic)}{3g}\right]$$

$$a_{21} = \left[B + \frac{\omega^2 R^5 k_2(cent, elastic)}{3g}\right]^{-1} \omega\left[C - A - \frac{\omega^2 R^5 k_2(cent, elastic)}{3g}\right]$$

$$a_{22} = -\left[B + \frac{\omega^2 R^5 k_2(cent, elastic)}{3g}\right]^{-1} \frac{\omega^2 R^5 k_{2,R}(0, cent)}{3g}$$

“the eigenvalues of the matrix A“

$$|A-\lambda I_2|=0 \quad \Leftrightarrow \quad \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix}=0 \quad \Leftrightarrow$$

$$(a_{11}-\lambda)(a_{22}-\lambda)-a_{12}a_{21}=0 \quad \Leftrightarrow$$

$$\lambda^2-\lambda(a_{11}+a_{22})+a_{11}a_{22}-a_{12}a_{21}=0 \quad \Leftrightarrow$$

$$\lambda^2-\lambda\, trA+\det A=0 \quad \Leftrightarrow$$

$$\lambda_{1,2}(A)=\frac{1}{2}trA\pm\frac{1}{2}\sqrt{(trA)^2-4\det A}$$

“special case: $a_{11}=0, \quad a_{22}=0, \quad A=B, \quad a_{12}=-a_{21}$“

$$\lambda_{12}(A)=\pm\sqrt{-\det A}=\pm a_{12}=\pm\left[A+\frac{\omega^2R^5k_2(cent,elastic)}{3g}\right]^{-1}\omega$$

$$+\left[A-C+\frac{\omega^2R^5k_2(cent,elastic)}{3g}\right]$$

Box 11

Polar motion equations in the time domain

"system of integro-differential equations of second order"

$$\dot{x} = Ax + f(x) + b$$

$$\ddot{x} = A\dot{x} + \dot{f} + \dot{b} = A(Ax + f(x) + b) + \dot{f} + \dot{b}$$

$$\ddot{x} = A^2 x + (Af + \dot{f}) + Ab + \dot{b}$$

$$\boxed{\ddot{x} - A^2 x - (Af + \dot{f}) = Ab + \dot{b}}$$

"special case

$$a_{11} = 0, \quad a_{22} = 0, \quad A = B, \quad a_{12} = -a_{21}\text{"}$$

$$-A^2 = -\begin{bmatrix} a_{12}a_{21} & 0 \\ 0 & a_{12}a_{21} \end{bmatrix} = \begin{bmatrix} a_{12}^2 & 0 \\ 0 & a_{12}^2 \end{bmatrix}$$

$$\lambda_{12}(-A^2) = a_{12}^2 = \frac{\omega^2 \left[A - C + \dfrac{\omega^2 R^5 k_2(cent, elastic)}{3g} \right]^2}{\left[A + \dfrac{\omega^2 R^5 k_2(cent, elastic)}{3g} \right]^2} \quad \in \mathbb{R}^+$$

"excited circular harmonic oscillator"

"general case: $a_{11} \neq 0, \quad a_{22} \neq 0, \quad a_{12} \neq a_{21}$"

$$\lambda_1 \neq \lambda_2 \quad \in \mathbb{R}^+$$

"excited elliptic harmonic oscillator"

Box 12

Principle of balance of Moment of Momentum

(Angular Momentum)

in a quasi-body fixed in frame of reference equation of length of day variation in the time domain.

"integro-differential equation of first order"

$$\left[C+\frac{4}{9}\frac{\omega^2 R^5}{g}k_2(cent, elastic)\right]\dot{x}_3+\frac{4}{9}\frac{\omega^2 R^5}{g}k_{2,R}(0, cent)x_3$$

$$+\int_0^t \dot{k}_2(t-t', cent)x_3(t')dt'+\frac{4}{9}\frac{\omega^2 R^5}{g}\int_0^t k_{2,R}(t-t', cent)x_3(t')dt'$$

$$=h(incr.torque, rel.ang.mom., tide, load, stress)$$

„system equation"

$$\dot{x}_3=a_{33}x_3+f_3(x_3)+b_3$$

$$a_{33}=-\left[C+\frac{4}{9}\frac{\omega^2 R^5}{g}k_2(cent, elastic)\right]^{-1}\frac{4}{9}\frac{\omega^2 R^5}{g}k_{2,R}(0, cent)x_3$$

$$f_3(x_3):=\left[C+\frac{4}{9}\frac{\omega^2 R^5}{g}k_2(cent, elastic)\right]^{-1}\int_0^t \dot{k}_2(t-t', cent)x_3(t')dt'$$

$$b_3:=\left[C+\frac{4}{9}\frac{\omega^2 R^5}{g}k_2(cent, elastic)\right]^{-1}h$$

$$\dot{f}_3 = f_{31} x_3 + \dot{f}_{32}$$

$$\dot{f}_{31} := \left[C + \frac{4}{9} \frac{\omega^2 R^5}{\gamma} k_2(cent, elastic) \right]^{-1} \dot{k}_2(0, cent)$$

$$\dot{f}_{32} := \left[C + \frac{4}{9} \frac{\omega^2 R^5}{g} k_2(cent, elastic) \right]^{-1} \int_0^t \ddot{k}_2(t - t', cent) x_3(t') dt'$$

„system equation: integro differential equation

of second order“

$$\ddot{x}_{33} + (f_{31} - a_{33}^2) x_3 - (a_{33} f_3 + \dot{f}_{32}) = a_{33} b_3 + \dot{b}_3$$

“excited, damped unharmonic (non-periodic) motion“

Box 18
Chandler Wobble "resonance"

$$s = \frac{\omega(3g\left(C-A\right) - k_2(cent)\omega^2 R^5)}{3Ag + k_2(cent)\omega^2 R^5}$$

$$\omega_{Ch} := \frac{1}{s} = \frac{3Ag + k_2(cent)\omega^2 R^5}{\omega(3g\left(C-A\right) - k_2(cent)\omega^2 R^5)}$$

Numeric example:

$$k_2(cent) = 0.3050488$$

$$R = 6.37100 \cdot 10^6 \ [m]$$

$$\omega = 0.72921 \cdot 10^{-4} \left[rad \cdot s^{-1}\right]$$

$$g = 6.67259 \cdot 10^{-11} \left[m^3 \cdot kg^1 \cdot s^{-2}\right]$$

$$A = 0.80670 \cdot 10^{38} \left[kg \cdot m^2\right]$$

$$B = 0.89380 \cdot 10^{38} \left[kg \cdot m^2\right]$$

$$\boxed{\omega_{Ch} = 439.2510 \left[days\right]}$$

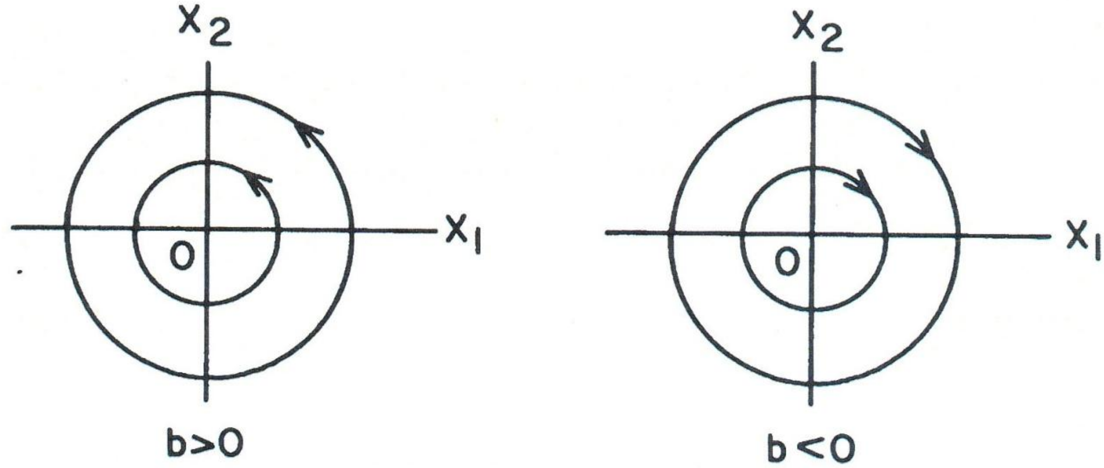


Figure 1: polar motion, elastic modeling

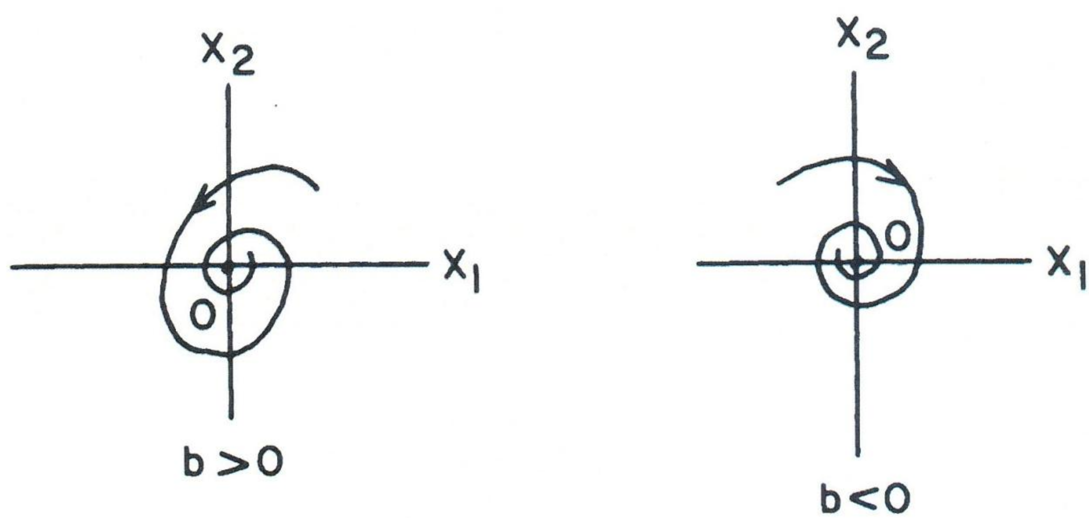


Figure 2: polar motion, viscoelastic modeling

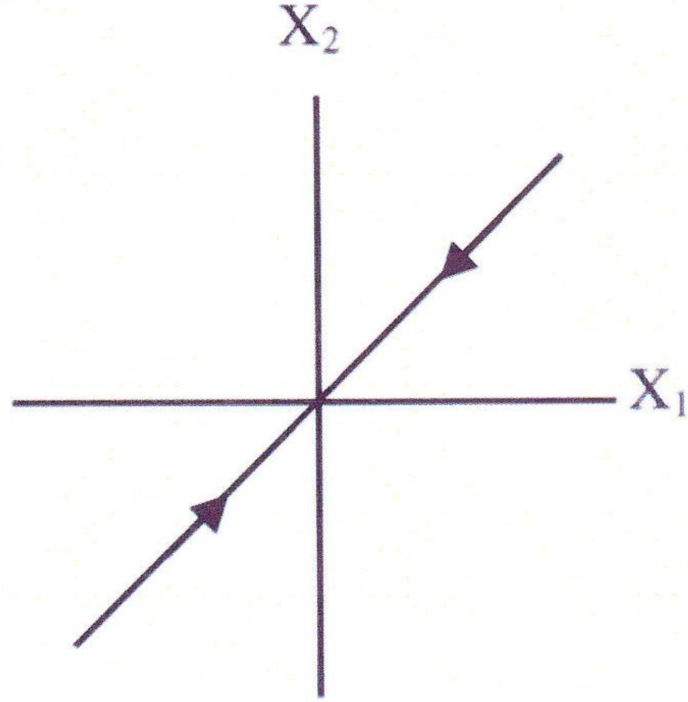


Figure 3: Lod, viscoelastic modeling

“stabl“ (“attractor“)

$$x_i = a_1 \cos(\omega_{Ch} t_i + \varphi_{11}) + a_2 \cos(\omega_{an} t_i + \varphi_{12}) + \alpha_1 t_i + \beta_1$$

$$y_i = b_1 \sin(\omega_{Ch} t_i + \varphi_{21}) + b_2 \sin(\omega_{an} t_i + \varphi_{22}) + \alpha_2 t_i + \beta_2$$

a_1=0.090638 [arcsec]

a_2=0.111806 [arcsec]

b_1=0.078297 [arcsec]

b_2=0.113356 [arcsec]

T_{an}=1.000141 [year]

T_{Ch}=1.174644 [year]

φ_{11}=5.278390 [rad]

φ_{12}=0.350699 [rad]

φ_{21}=5.244691 [rad]

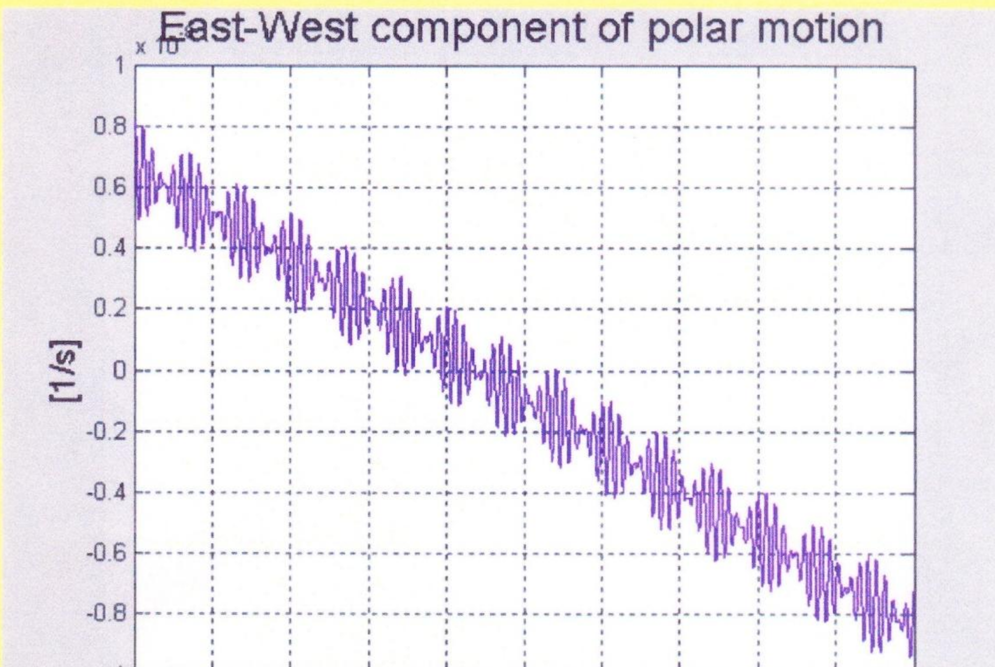
φ_{22}=0.330228 [rad]

α_1=0.000909 [arcsec/year]

β_1=0.002635 [arcsec]

α_2=0.004333 [arcsec/year]

β_2=0.102778 [arcsec]



(**H. Schuh , S. Nagel , T. Seitz** (2001)**:** *Linear drift and periodic variations observed in long time series of polar motion*, Journal of Geodesy, Vol. 74, Issue 10, pp 701-710)

Summary

The Liouville perturbation theory of the Euler dynamical equations of angular momentum of the Earth considered as a deformable body leads to a first order inhomogeneous system of integro-differential equations which are classified in terms of system theory. With respect to a viscoelastic Earth model of homogeneous spherical shells the spectrum of the Liouville operator is analyzed. Following a proposal of M. Schneider (Proc. Bundesamt für Karthographie und Geodäsie 5, pp. 28-33, Frankfurt 1999) the first order system is differentiated to a second order system and being alternatively classified as a second order inhomogeneous system of integro-differential equations. It leads to the interpretation that the characteristic equations of Polar Motion represent an excited, coupled, damped approximately elliptic oscillator, while the characteristic equation of length-of-Day variation documents an excited, damped non-periodic motion. Solutions are represented both in the Laplace domain as well as in the Fourier domain. New solutions are presented in the dynamical waveled domain as well as in the fractal domain, tentatively.