The boundary elements formulation of Molodensky's problem: new ideas from the old book of physical geodesy

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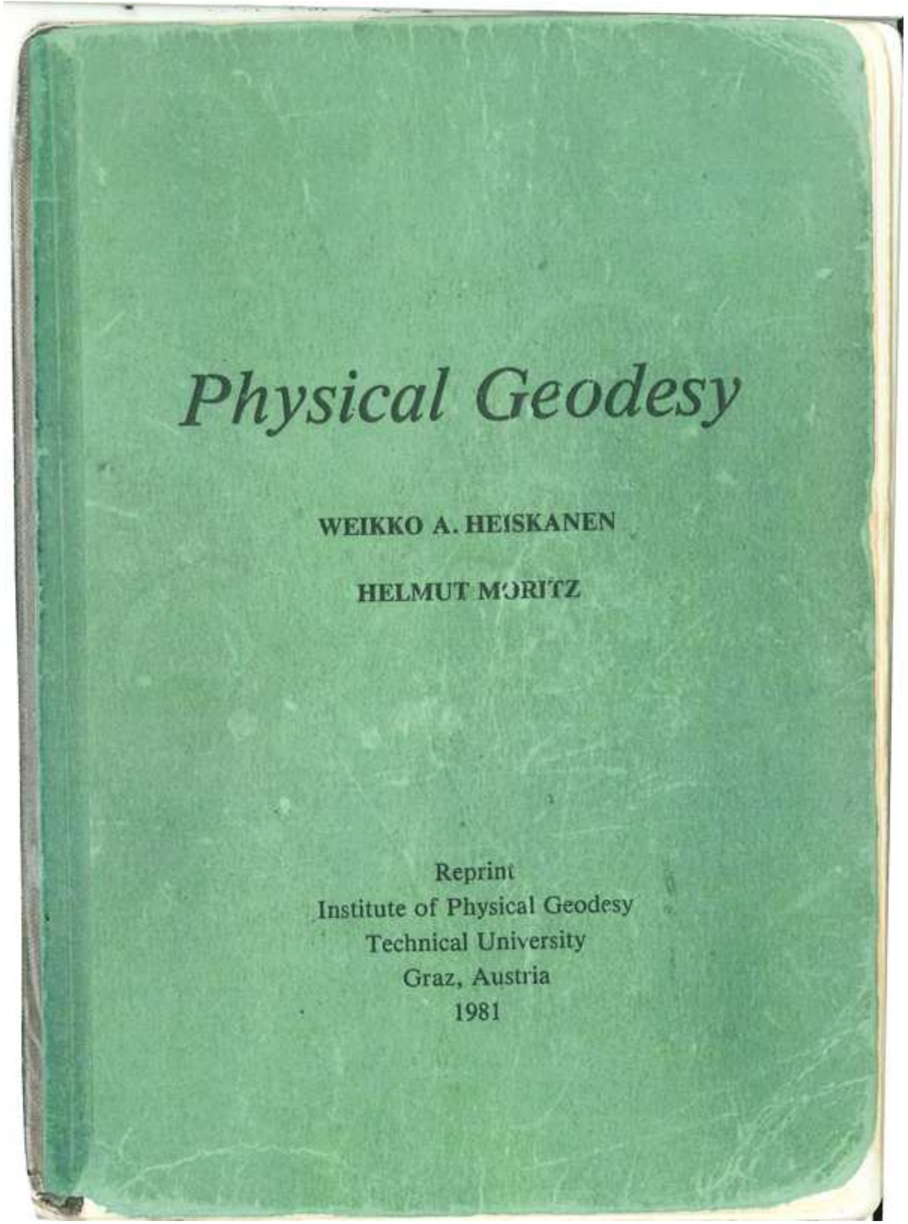
Helmut Moritz has been

my teacher of Geodesy

This is the book
where I studied
and I continue ...

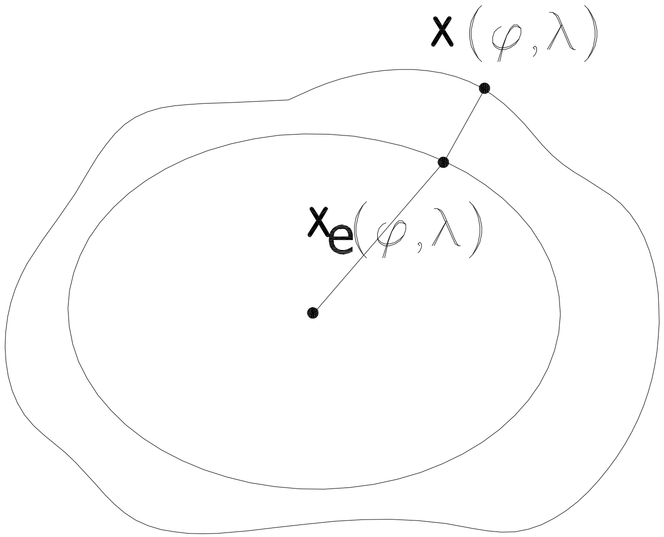


The scalar Molodensky problem is to find

$W(\mathbf{x})$ gravity potential

$S \equiv \{\mathbf{x}(\lambda, \varphi\}$ Earth surface $\mathbf{x}(\lambda, \varphi) = \mathbf{x}_e(\lambda, \varphi) + h(\lambda, \varphi)\nu(\lambda, \varphi)$

from $W(\lambda, \varphi) \equiv W|_S$, $g(\lambda, \varphi)|\nabla W||_S$



S'

$$\begin{cases} \Delta\{W - \frac{1}{2}\omega^2(x^2 + y^2)\} = 0 \\ W|_S = W(\lambda, \varphi) \\ |\nabla W||_S = g(\lambda, \varphi) \end{cases}$$

The scalar Molodensky problem is to find

$$T(\mathbf{x}) \text{ anomalous potential}$$

$$\zeta = \frac{T}{\gamma} \text{ height anomaly}$$

from the BVP formulated on a telluroid

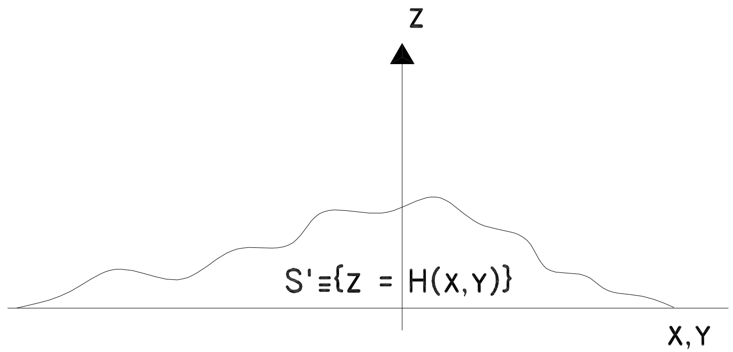
$$S \equiv \{U(\widetilde{h}) \equiv W(h)\}$$

from the solution of the oblique derivative BVP

$$\begin{cases} \Delta T = 0 \\ \left(-\frac{\partial T}{\partial h} - \frac{\frac{\partial \gamma}{\partial h}}{\gamma} T\right)\Big|_S = \Delta g \end{cases}$$

(we ignore here the asymptotic conditions, since we watn to go to a simple planar approximation)

The planar Molodensky problem in planar approximation is to find $T(\mathbf{x})$ such that



$$\begin{cases} \Delta T = 0 & \{z \geq H(x,y) \equiv H(\xi)\} \\ \frac{\partial T}{\partial z}\Big|_S \equiv T'\big|_S = -\Delta g & \{z = H(x,y)\} \\ T \to 0 \ , & z \to \infty \end{cases}$$

Notice the use of the notation

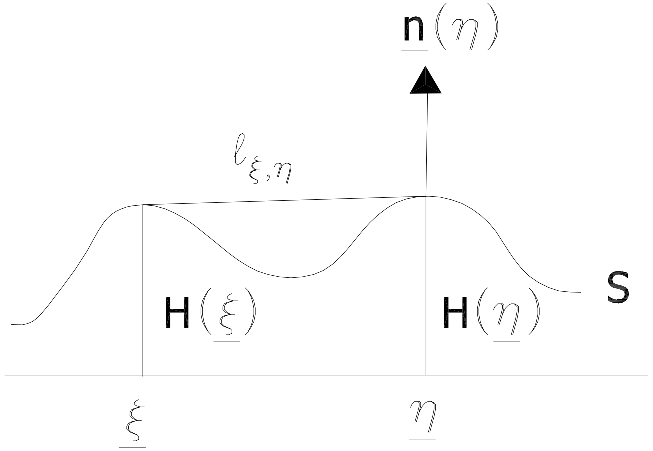
$$\xi = \begin{vmatrix} x \\ y \end{vmatrix}$$

$$\nabla_0 T = \mathbf{e}_x \frac{\partial T}{\partial X} + \mathbf{e}_y \frac{\partial}{\partial y} T = \nabla_\xi T$$

$$T' = \frac{\partial T}{\partial Z}$$

The third Green identity - or better its limit when $z \to H(\xi)$.

$$T(\xi, H(\xi)) = \frac{1}{2\pi} \int_S \{T(\eta, H(\eta)) \frac{\partial}{\partial n_\eta} \, \frac{1}{\ell_{\xi\eta}} - \frac{\partial T}{\partial n_\eta}(\eta, H(\eta)) \frac{1}{\ell_{\xi\eta}}\} dS_\eta$$



Note

$$\ell_{\xi\eta} = [|\xi - \eta|^2 + (H(\xi) - H(\eta))^2]^{\frac{1}{2}}$$

$$\frac{\partial T(\eta, H)}{\partial n_\eta} = \mathbf{n}_\eta \cdot (\nabla_0 T + \mathbf{e}_z T')(\eta, H(\eta))$$

first the 3D gradient, then the trace on S!

$$\frac{\partial}{\partial n_\eta} \frac{1}{\ell_{\xi,\eta}} = \mathbf{n}_\eta \cdot \frac{(\xi - \eta) + (H(\xi) - H(\eta))\mathbf{e}_z}{\ell^3_{\xi\eta}}$$

Important: if we introduce the trace and the Neumann operators

$$\Gamma_z\{T(\mathbf{x})\} \equiv \Gamma_z\{T(\xi,x)\} = T(\xi,H(\xi)) \ ; \ N_{\xi,z}\{T(\mathbf{x})\} \equiv N_{\xi,z}\{T(\xi,z\} = \mathbf{n}(\xi)\cdot\nabla T(\xi,z)$$

noting that

$$\ell_{\mathbf{x},\mathbf{y}} = \ell(\xi,z\ ;\ \eta,z') = [|\xi-\eta|^2 + (z-z')^2]^{\frac{1}{2}}$$

we can write the third Green identity more precisely as

$$\Gamma_z(T(\xi,z)) = \frac{1}{2\pi}\int_S \{\Gamma_{z'}(T(\eta,z')\Gamma_{z'}[N_{\eta,z'}(\ell^{-1}_{\mathbf{x},\mathbf{y}})] +$$

$$-\Gamma_{z'}[N_{\eta,z'}(T(\eta,z')]\Gamma_{z'}(\ell^{-1}_{\mathbf{x},\mathbf{y}}\}dS_y$$

Boundary elements: notice that when

$$-\Gamma_{z'} N_{\eta,z'}(T(\eta, z') \equiv -T_n(\eta, H(\eta)) = g(\eta)$$

is known one can compute

$$h(\xi) = -\frac{1}{2\pi}\int g(\eta)\Gamma_z\Gamma_{z'}(\ell^{-1}_{\mathbf{x},\mathbf{y}})dS_\eta$$

and the third Green identity becomes the integral equation

$$f(\xi) - \frac{1}{2\pi}\int f(\eta)\frac{\mathbf{n}(\eta)\cdot(\mathbf{x}-\mathbf{y})}{\ell^3_{\mathbf{x},\mathbf{y}}}dS_\eta = h(\xi)$$

$$f(\xi) = T(\xi, H(\xi))$$

the solution of which gives directly T on S.

Can we write the B.E. integral equation for the Molodensky problem?

Yes but we need several steps and caution

- Put as before

$$f(\xi) = T(\xi, H(\xi)) = \Gamma_z(T)$$
$$-\Delta g(\xi) = -T'(\xi, H(\xi)) = -\Gamma_z(T')$$

- define $\mathbf{v}(\xi) = \nabla_0 H(\xi)$

- note that

$$\nabla_0 \Gamma(T) \equiv \nabla_0 f = \Gamma \nabla_0 (T) + \Gamma(T') \nabla_0 H$$

so that

$$\Gamma \nabla_0 (T) = \nabla_0 f + \Delta g \mathbf{v}$$

- moreover we have the following geometric identities

$$\mathbf{n} = \cos I = (-\mathbf{v} + \mathbf{e}_z)$$

$$\cos I = [|\mathbf{v}|^2 + 1]^{-\frac{1}{2}} \ ; \quad \text{tg } I = |\mathbf{v}|$$

$$\cos I dS_\eta = dS_0 = \text{ area element in the horizontal plane}$$

-

$$\begin{aligned} \Gamma N(T) &= \Gamma[(-\mathbf{v} + \mathbf{e}_z) \cdot (\nabla_0 T + T' \mathbf{e}_z)] \cos I = \\ &= [-\mathbf{v} \cdot \Gamma \nabla_0 T - \Delta g] \cos I = \\ &= [-\mathbf{v} \cdot \nabla_0 f - \Delta g |\mathbf{v}|^2 - \Delta g] \cos I = \\ &= [-\mathbf{v} \cdot \nabla_0 f - (1 + \text{ tg}^2 I) \Delta g] \cos I \end{aligned}$$

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$$-\int \Gamma_{z'}[N(T)]\Gamma_z\Gamma_{z'}\ell_{\mathbf{x},\mathbf{y}}^{-1}dS_\eta = \int \mathbf{v}(\eta)\cdot\nabla_0 f(\eta)\ell_{\xi,\eta}^{-1}dS_0 +$$

$$+\int (I+\ \mathrm{tg}^2 I)\Delta g(\eta)\ell_{\xi,\eta}^{-1}dS_0 \equiv \int \mathbf{v}(\eta)\cdot\nabla_0 f(\eta)\ell_{\xi,\eta}^{-1}dS_0 + h(\xi)$$

- crucial step is the integration by parts

$$\int \mathbf{v}(\xi)\cdot\nabla_0 f(\xi)\ell_{\xi,\eta}^{-1}dS_0 = -\int \nabla_0\cdot\mathbf{v}(\eta)f(\eta)\ell_{\xi,\eta}^{-1}dS_0 +$$

$$-\int \mathbf{v}(\eta)\cdot\frac{(\xi-\eta)+(H(\xi)-H(\eta))\mathbf{v}(\eta)}{\ell_{\xi\eta}^3}f(\eta)dS_0 = (\nabla_0\cdot\mathbf{v}(\eta) = \Delta_0 H(\eta))$$

$$-\int \frac{\Delta_0 H(\eta)f(\eta)}{\ell_{\xi,\eta}}dS_0 - \int \frac{\mathbf{v}(\eta)\cdot(\xi-\eta)+(H(\xi)-H(\eta))|\mathbf{v}(\eta)|^2}{\ell_{\xi\eta}^3}f(\eta)dS_0$$

Singular integral: compute in Cauchy principal part

- Similarly

$$
\begin{aligned}
\Gamma_z\Gamma_{z'}\{N_{\eta,z'}(\ell^{-1}_{\mathbf{x},\mathbf{y}})\} &= \Gamma_z\Gamma_{z'}\{\mathbf{n}(\eta)\cdot\frac{(\xi-\eta)+(z-z')\mathbf{e}_z}{\ell^3_{\mathbf{x},\mathbf{y}}}\} = \\
&= [(-\mathbf{v}(\eta)+\mathbf{e}_z)\cdot\frac{(\xi-\eta)+(H(\xi)-H(\eta)\mathbf{e}_z}{\ell^3_{\xi,\eta}}]\cos I \\
&= \frac{-\mathbf{v}(\eta)\cdot(\xi-\eta)+(H(\xi)-H(\eta))}{\ell^3_{\xi,\eta}}\cos I
\end{aligned}
$$

so that

$$
\int f(\eta)\Gamma_z\Gamma_{z'}[N(\ell^{-1}_{\mathbf{x},\mathbf{y}})]dS_y = \int f(\eta)\frac{-\mathbf{v}(\eta)\cdot(\xi-\eta)+(H(\xi)-H(\eta))}{\ell^3_{\xi,\eta}}dS_0
$$

- Summarizing

$$2\pi f(\xi) = h(\xi) - \int \frac{\Delta H(\eta) f(\eta)}{\ell_{\xi,\eta}} dS_0 +$$
$$+ \int f(\eta) \frac{-2\mathbf{v}(\eta)\cdot(\xi-\eta) + (H(\xi) - H(\eta))(I - |\nu(\eta)|^2)}{\ell^3_{\xi,\eta}} dS_0$$

This is the Boundary Element equation for Molodensky's problem: it is a singular integral equation, as unavoidable for an oblique derivative problem.

A last comment

Since Fredholm alternative holds, the uniqueness of the solution is particularly important.

This can be achieved directly by the Cartesian version of the energy integral.

From

$$\begin{aligned}\nabla \cdot (T' \nabla T) &= \frac{\partial}{\partial z} \nabla T \cdot \nabla T = \\ &= \frac{1}{2} \frac{\partial}{\partial z} |\nabla T|^2\end{aligned}$$

one has

$$-\int_S T' T_n dS = \int dS_0 \int_{H(\xi)}^{+\infty} dz \frac{1}{2} \frac{\partial}{\partial z} |\nabla T|^2 = -\frac{1}{2} \int dS_0 |\nabla T|_S^2$$

So, putting $J = (\cos I)^{-1}$ and $J_+ = \max_S (\cos I)^{-1}$,

$$\int dS_0 |\nabla T|_S^2 = 2 \int T' T_n J dS_0 \leq 2J_+ [\int \Delta g^2 dS_0]^{\frac{1}{2}} [\int |T_n|^2 dS_0]^{\frac{1}{2}} \leq$$

$$\leq 2J_+ [\int \Delta g^2 dS_0]^{\frac{1}{2}} [\int |\nabla T|_S^2 dS_0]$$

Simplyfing we get

$$[\int dS_0 |\nabla T|_S^2]^{\frac{1}{2}} \leq 2J_2 [\int \Delta g^2 dS_0]^{\frac{1}{2}}$$

that guarantees uniqueness and stability of the solution!