# Friedrich Robert Helmert:
# seine Lösung des Anholonomitäts-Problem der Geodäsie
# oder
# warum ist Geodäsie physikalisch?
# Von Gauss – Listing über Bruns zu Molodenski:
# geometrische und physikalische Höhen und Höhensysteme:
# global versus lokal


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# *Friedrich Robert Helmert*

geboren in Freiberg in Sachsen
Am 31. Juli 1843
(Gedenktafel 1913, erneuert 1967)

*geodätische Vorläufer*: C.F. Gauss, Soldner (Bayern), Bohnenberger (Württemberg), Baeyer (Preußen), Gerling (Hessen), Hansen (Sachsen-Coburg)

*(i)* Bürgerschule in Freiberg

(v) Ergänzungsstudium Mathe/Physik
in Leipzig (2 Jahre)

(vi) 1868: Dissertation

„Studium über rationelle Vermessungen im Gebiet der Höheren Geodäsie“

(vii) 1869: Observator Sternwarte Hamburg

1874: erste Publikation

„Vermessung und rechnerische Ausgleichung eines Sternhaufens“

(viii) 1870: Berufung auf die ordentliche Professur „Geodäsie“ an der Polytechnischen Schule/Universität Aachen mit 29 Jahren!

*2 fundamentale Werke:*

A) Ausgleichungsrechnung nach der Methode der kleinsten Quadrate mit Anwendungen auf die Geodäsie und die Theorie: 1872

B) Die mathematischen und physikalischen Theorien der

## *Helmert‘s Arbeitsthemen:*

- *Math. Statistik und Ausgleichungsrechnung*

+ HELMERT TRANSFORMATION

überbestimmtes Ausgleichungsproblem:
Translation Rotation, Maßstab, „Lineare Gruppe“

+ CHI-QUADRAT VERTEILUNG

+ GAUSS-HELMERT MODEL

Bedingungsgleichungen mit Unbekannten

+ Var-Kovarianz Schätzung

„Varianz Komponenten-Schätzer, welche positiv sind“

+ CHOLESKI FAKTORISIERUNG

+ STRENGE AUSGLEICHUNG des EUROPÄISCHEN DREIECKSNETZES

• *Physikalische Geodäsie*

+ HELMERT Projektion

+ HELMERT‘s ELLIPSOID - Übergang

+ HELMERT NIVEAUELLIPSOID

+ HELMERT HÖHEN

+ HELMERT LOTABWEICHUNGEN „Defintion“

+ POLHÖHEN Änderungen

+ GEODYNAMIK

„HELMERT“: HOLONOM versus ANHOLONOM:

? Warum ist Geodäsie physikalisch ?

F.R. HELMERT hatte auf Rat seines Doktorvaters zwei Jahre Physik in Leipzig studiert:

! FROBENIUS LEMMA !

„integrierende Funktionen“

$$\oint dW = -\oint \Gamma \, đH = 0 \quad \text{versus} \quad \oint đH \quad : \quad \text{M. PLANCK}$$

„RING-INTEGRAL“: geschlossener Weg

„W ist das GAUSS-GREEN Potential“

CARTAN Differential-Geometry: „CARTAN calculus“

In einem mitrotierenden Koordinaten-System gibt es 4 Typen von Kräften:

(i) Gravitation, (ii) Zentrifugalkraft: KONSERVATIV

(iii) EULER („Drehimpuls“: Polbewegung, Tageslängenänderung oder Präzession/Nutation) NICHT-KONSERVATIV

(iv) CORIOLIS („Strömungen“): NICHT-KONSERVATIV

(ii) $\mathrm{div}\,\mathrm{grad}\, W = -4\pi G\rho + 2\Omega^2$

KONSERVATIV: (i) $\mathrm{grad}\, W = \mathrm{grad}\, U + \mathrm{grad}\, V$

NICHT-KONSERVATIV: (iii) $\mathrm{rot}\, \Gamma = 2\Omega\bullet$ (starrer Körper, der rotiert)

BASIS: DIFFERENTIAL FORMEN

„EXTERIOR CALCULUS“: ELIE CARTAN

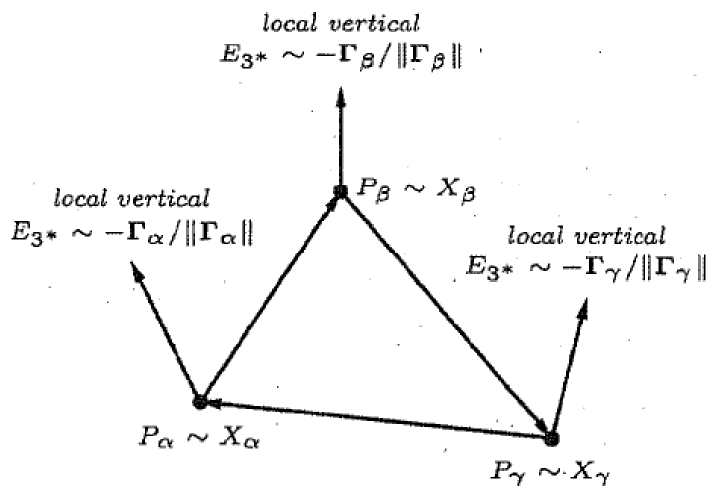
# „HELMERT“:

2 Beispiele

ZfV 122 (1987) 413 – 424:

„Der Einfluß der Lotrichtung auf lokale geodätische Netze“

BEISPIEL:




Triangular network $\{P_\alpha, P_\beta, P_\gamma | \mathbf{O}\}$, placement vectors at the origin $\mathbf{O}$, local verticals $E_3(P_\alpha)$, $E_3(P_\beta)$, $E_3(P_\gamma)$, $\boldsymbol{\Gamma}_\alpha$, $\boldsymbol{\Gamma}_\beta$, $\boldsymbol{\Gamma}_\gamma$ local gravity vectors

$$\overset{\alpha}{X}_{\alpha\beta} := \overset{\alpha}{X}_\beta - \overset{\alpha}{X}_\alpha, \ldots, \overset{\alpha}{Z}_{\gamma\alpha} := \overset{\alpha}{Z}_\alpha - \overset{\alpha}{Z}_\gamma\,.$$

$$\overset{\alpha}{X}_{\alpha\beta} + \overset{\alpha}{X}_{\beta\gamma} + \overset{\alpha}{X}_{\gamma\alpha} = 0$$

$$\overset{\alpha}{Y}_{\alpha\beta} + \overset{\alpha}{Y}_{\beta\gamma} + \overset{\alpha}{Y}_{\gamma\alpha} = 0$$

$$\overset{\alpha}{Z}_{\alpha\beta} + \overset{\alpha}{Z}_{\beta\gamma} + \overset{\alpha}{Z}_{\gamma\alpha} = 0$$

Holonomity condition in terms of relative coordinates in a fixed reference system, fixed to the reference point $P_\alpha$

# BEISPIEL:

$$
\begin{tikzcd}
 & F^\circ \arrow[dl, "\mathbf{R}_E(\Lambda_T,\Phi_T,0)(P_\alpha)"'] \arrow[dr, "\mathbf{R}_E(\Lambda_T,\Phi_T,0)(P_\gamma)"] & \\
E_*(P_\alpha) \arrow[rr] & & E_*(P_\gamma)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & F^\circ \arrow[dl, "\mathbf{R}_E(\Lambda_T,\Phi_T,0)(P_\alpha)"'] \arrow[dr, "\mathbf{R}_E(\Lambda_T,\Phi_T,0)(P_\beta)"] & \\
E_*(P_\alpha) \arrow[rr] & & E_*(P_\beta)
\end{tikzcd}
$$

Commutative diagrams: *moving horizon reference systems* $E_*$ versus *fixed equatorial reference systems* $F^\circ$

# E. GRAFAREND: ZfV 122 (1987) 413-424

*"anholonomity condition"*

$$\overset{\alpha}{X}_{\alpha\beta} := \overset{\alpha}{X}_{\beta} - \overset{\alpha}{X}_{\alpha}, \ldots, \overset{\alpha}{Z}_{\gamma\alpha} := \overset{\alpha}{Z}_{\alpha} - \overset{\alpha}{Z}_{\gamma}$$

$$\overset{\alpha}{X}_{\alpha\beta} + \overset{\beta}{X}_{\beta\gamma} + \overset{\gamma}{X}_{\gamma\alpha} \neq 0 \quad \text{(misclosure)}$$

$$\overset{\alpha}{Y}_{\alpha\beta} + \overset{\beta}{Y}_{\beta\gamma} + \overset{\gamma}{Y}_{\gamma\alpha} \neq 0 \quad \text{(misclosure)}$$

$$\overset{\alpha}{Z}_{\alpha\beta} + \overset{\beta}{Z}_{\beta\gamma} + \overset{\gamma}{Z}_{\gamma\alpha} \neq 0 \quad \text{(misclosure)}$$

*"representation of the base vectors in the horizon reference frame $E^*$"*

$$\mathbf{E}_{1*} X_{\alpha\beta} + \mathbf{E}_{2*} Y_{\alpha\beta} + \mathbf{E}_{3*} Z_{\alpha\beta}$$

*"Direct and inverse transformation of Cartesian coordinates into spherical coordinates"*

(horizontal coordinate $H_{\alpha\beta}$, vertical coordinate $V_{\alpha\beta}$, distance $S$, azimuth $A_{\alpha\beta}$, vertical angle $B_{\alpha\beta}$, horizontal orientiation unknown)

$$X_{\alpha\beta} = S_{\alpha\beta} \cos A_{\alpha\beta} \cos B_{\alpha\beta},$$
$$Y_{\alpha\beta} = S_{\alpha\beta} \sin A_{\alpha\beta} \cos B_{\alpha\beta},$$
$$Z_{\alpha\beta} = S_{\alpha\beta} \sin B_{\alpha\beta}$$
$$A_{\alpha\beta} := H_{\alpha\beta} + \Sigma_{\alpha} = \arctan\left(Y_{\alpha\beta}/X_{\alpha\beta}\right),$$
$$B_{\alpha\beta} := V_{\alpha\beta} = \arctan\left(Z_{\alpha\beta}/\sqrt{X_{\alpha\beta} + Y_{\alpha\beta}}\right)$$
$$S_{\alpha\beta} := \sqrt{X^2_{\alpha\beta} + Y^2_{\alpha\beta} + Z^2_{\alpha\beta}}$$

*Point transformation*
$\mathbf{E}^{\alpha}_{*} \to \mathbf{E}^{\beta}_{*}$ *and* $\mathbf{E}^{\beta}_{*} \to \mathbf{E}^{\gamma}_{*}$

*"Euler angles"*

$$\mathbf{E}^{\alpha}(P_{\alpha}) \to \mathbf{E}^{\beta}(P_{\beta}):$$
$$\mathbf{E}^{\alpha}(P_{\alpha})\mathbf{R}_E(\Lambda_{\alpha}, \Phi_{\alpha}, 0)\mathbf{R}^T_E(\Lambda_{\beta}, \Phi_{\beta}, 0)$$
$$\Delta\Lambda := \Lambda_{\beta} - \Lambda_{\alpha}, \quad \Delta\Phi := \Phi_{\beta} - \Phi_{\alpha}$$
$$\mathbf{R}_E(\Lambda_{\alpha}, \Phi_{\alpha}, 0)\mathbf{R}^T_E(\Lambda_{\beta}, \Phi_{\beta}, 0)$$
$$\doteq \begin{bmatrix} 1 & -\Delta\Lambda \sin\Phi_{\alpha} & \Delta\Phi \\ \Delta\Lambda \sin\Phi_{\alpha} & 1 & \Delta\Lambda \cos\Phi_{\alpha} \\ \Delta\Phi & -\Delta\Lambda \cos\Phi_{\alpha} & 1 \end{bmatrix}$$

*"antisymmetric matrix* $\mathbf{A}$*"*

$$-\begin{bmatrix} \overset{\beta}{X}_{\beta\alpha} \\ \overset{\beta}{Y}_{\beta\alpha} \\ \overset{\beta}{Z}_{\beta\alpha} \end{bmatrix} = (\mathbf{I} - \mathbf{A}^T) \begin{bmatrix} \overset{\alpha}{X}_{\alpha\beta} \\ \overset{\alpha}{Y}_{\alpha\beta} \\ \overset{\alpha}{Z}_{\alpha\beta} \end{bmatrix}$$

*analogue formulae for*

$$X_{\gamma\alpha}, \ldots, Z_{\gamma\alpha}$$

*analogue formulae for*

$$A_{\beta\alpha},\ B_{\beta\alpha},\ S_{\beta\alpha},\ \ldots,\ A_{\gamma\alpha},\ B_{\gamma\alpha},\ S_{\gamma\alpha}$$

Anholonomity in a moving frame at points $\{P_{\alpha}, P_{\beta}, P_{\gamma}\}$

# Erstes Beispiel

$$\overset{\alpha}{X}_{\alpha\beta} = +30\ m, \quad \overset{\alpha}{X}_{\beta\gamma} = +50\ m, \quad \overset{\alpha}{X}_{\gamma\alpha} = -80\ m$$

$$\overset{\alpha}{Y}_{\alpha\beta} = +30\ m, \quad \overset{\alpha}{Y}_{\beta\gamma} = -50\ m, \quad \overset{\alpha}{Y}_{\gamma\alpha} = +20\ m$$

$$\overset{\alpha}{Z}_{\alpha\beta} = +5\ m, \quad \overset{\alpha}{Z}_{\beta\gamma} = +15\ m, \quad \overset{\alpha}{Z}_{\gamma\alpha} = -20\ m$$

$$\Lambda_{\alpha\beta} = 1'' \sim 4.85 \cdot 10^{-6}\ \text{RAD}, \quad \Phi_{\alpha\beta} = -0.5'' \sim -2.42 \cdot 10^{-6}\ \text{RAD}$$

$$\Lambda_{\alpha\gamma} = -1'' \sim -4.85 \cdot 10^{-6}\ \text{RAD}, \quad \Phi_{\alpha\gamma} = -2.5'' \sim -12.12 \cdot 10^{-6}\ \text{RAD}$$

$$\Phi_\alpha = 48.783^\circ$$

25 meter local network

$$\begin{bmatrix} \overset{\beta}{X}_{\beta\gamma} \\ \overset{\beta}{Y}_{\beta\gamma} \\ \overset{\beta}{Z}_{\beta\gamma} \end{bmatrix} = \begin{bmatrix} 1 & +\Lambda_{\alpha\beta}\sin\Phi_\alpha & +\Phi_{\alpha\beta} \\ -\Lambda_{\alpha\beta} & 1 & -\Lambda_{\alpha\beta}\cos\Phi_\alpha \\ -\Phi_{\alpha\beta} & +\Lambda_{\alpha\beta}\cos\Phi\alpha & 1 \end{bmatrix} \begin{bmatrix} \overset{\alpha}{X}_{\beta\gamma} \\ \overset{\alpha}{Y}_{\beta\gamma} \\ \overset{\alpha}{Z}_{\beta\gamma} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & +3.65\cdot 10^{-6} & -2.42\cdot 10^{-6} \\ -3.65\cdot 10^{-6} & 1 & -3.19\cdot 10^{-6} \\ +2.42\cdot 10^{-6} & +3.19\cdot 10^{-6} & 1 \end{bmatrix} \begin{bmatrix} +50\ m \\ -50\ m \\ +15\ m \end{bmatrix}$$

$$\begin{bmatrix} \overset{\gamma}{X}_{\gamma\alpha} \\ \overset{\gamma}{Y}_{\gamma\alpha} \\ \overset{\gamma}{Z}_{\gamma\alpha} \end{bmatrix} = \begin{bmatrix} 1 & +\Lambda_{\alpha\gamma}\sin\Phi_\alpha & +\Phi_{\alpha\gamma} \\ -\Lambda_{\alpha\gamma} & 1 & -\Lambda_{\alpha\gamma}\cos\Phi_\alpha \\ -\Phi_{\alpha\gamma} & +\Lambda_{\alpha\gamma}\cos\Phi\alpha & 1 \end{bmatrix} \begin{bmatrix} \overset{\alpha}{X}_{\gamma\alpha} \\ \overset{\alpha}{Y}_{\gamma\alpha} \\ \overset{\alpha}{Z}_{\gamma\alpha} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & +-3.65\cdot 10^{-6} & -12.12\cdot 10^{-6} \\ +3.65\cdot 10^{-6} & 1 & +3.19\cdot 10^{-6} \\ +12.12\cdot 10^{-6} & -3.19\cdot 10^{-6} & 1 \end{bmatrix} \begin{bmatrix} -80\ m \\ +20\ m \\ -20\ m \end{bmatrix}$$

25 meter local network, detailed computation

$$\overset{\beta}{X}_{\beta\gamma} = +50\ m - 0.22\ mm, \quad \overset{\gamma}{X}_{\gamma\alpha} = -80\ m + 0.17\ mm$$

$$\overset{\beta}{Y}_{\beta\gamma} = -50\ m - 0.23\ mm, \quad \overset{\gamma}{Y}_{\gamma\alpha} = +20\ m - 0.36\ mm$$

$$\overset{\beta}{Z}_{\beta\gamma} = +15\ m - 0.04\ mm, \quad \overset{\gamma}{Z}_{\gamma\alpha} = -20\ m - 1.03\ mm$$

$$\overset{\alpha}{X}_{\alpha\beta} + \overset{\beta}{X}_{\beta\gamma} + \overset{\gamma}{X}_{\gamma\alpha} \neq 0 : \quad -0.22\ mm + 0.17\ mm = -0.05\ mm$$

$$\overset{\alpha}{Y}_{\alpha\beta} + \overset{\beta}{Y}_{\beta\gamma} + \overset{\gamma}{Y}_{\gamma\alpha} \neq 0 : \quad -0.23\ mm - 0.35\ mm = -0.59\ mm$$

$$\overset{\alpha}{Z}_{\alpha\beta} + \overset{\beta}{Z}_{\beta\gamma} + \overset{\gamma}{Z}_{\gamma\alpha} \neq 0 : \quad -0.04\ mm - 1.03\ mm = -1.07\ mm$$

25 meter local network, misclosures

# E. GRAFAREND: ZfV 122 (1987) 413-424

## Zweites Beispiel

$$\overset{\alpha}{X}_{\alpha\beta} = +500\ m, \quad \overset{\alpha}{X}_{\beta\gamma} = +800\ m, \quad \overset{\alpha}{X}_{\gamma\alpha} = -1300\ m$$
$$\overset{\alpha}{Y}_{\alpha\beta} = +500\ m, \quad \overset{\alpha}{Y}_{\beta\gamma} = -800\ m, \quad \overset{\alpha}{Y}_{\gamma\alpha} = +300\ m$$
$$\overset{\alpha}{Z}_{\alpha\beta} = +50\ m, \quad \overset{\alpha}{Z}_{\beta\gamma} = +150\ m, \quad \overset{\alpha}{Z}_{\gamma\alpha} = -200\ m$$
$$\Lambda_{\alpha\beta} = 25'' \sim 12.12 \cdot 10^{-5}\ \text{RAD}, \quad \Phi_{\alpha\beta} = -15'' \sim -7.27 \cdot 10^{-5}\ \text{RAD}$$
$$\Lambda_{\alpha\gamma} = -15'' \sim -7.27 \cdot 10^{-5}\ \text{RAD}, \quad \Phi_{\alpha\gamma} = -45'' \sim -2.18 \cdot 10^{-4}\ \text{RAD}$$
$$\Phi_{\alpha} = 48.783^\circ$$

500 meter local network

$$\begin{bmatrix} \overset{\beta}{X}_{\beta\gamma} \\ \overset{\beta}{Y}_{\beta\gamma} \\ \overset{\beta}{Z}_{\beta\gamma} \end{bmatrix} = \begin{bmatrix} 1 & +\Lambda_{\alpha\beta}\sin\Phi_\alpha & +\Phi_{\alpha\beta} \\ -\Lambda_{\alpha\beta} & 1 & -\Lambda_{\alpha\beta}\cos\Phi_\alpha \\ -\Phi_{\alpha\beta} & +\Lambda_{\alpha\beta}\cos\Phi\alpha & 1 \end{bmatrix} \begin{bmatrix} \overset{\alpha}{X}_{\beta\gamma} \\ \overset{\alpha}{Y}_{\beta\gamma} \\ \overset{\alpha}{Z}_{\beta\gamma} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & +9.12\cdot 10^{-5} & -7.27\cdot 10^{-5} \\ -9.12\cdot 10^{-5} & 1 & -7.99\cdot 10^{-5} \\ +7.27\cdot 10^{-5} & +7.99\cdot 10^{-5} & 1 \end{bmatrix} \begin{bmatrix} +800\ m \\ -800\ m \\ +150\ m \end{bmatrix}$$

$$\begin{bmatrix} \overset{\gamma}{X}_{\gamma\alpha} \\ \overset{\gamma}{Y}_{\gamma\alpha} \\ \overset{\gamma}{Z}_{\gamma\alpha} \end{bmatrix} = \begin{bmatrix} 1 & +\Lambda_{\alpha\gamma}\sin\Phi_\alpha & +\Phi_{\alpha\gamma} \\ -\Lambda_{\alpha\gamma} & 1 & -\Lambda_{\alpha\gamma}\cos\Phi_\alpha \\ -\Phi_{\alpha\gamma} & +\Lambda_{\alpha\gamma}\cos\Phi\alpha & 1 \end{bmatrix} \begin{bmatrix} \overset{\alpha}{X}_{\gamma\alpha} \\ \overset{\alpha}{Y}_{\gamma\alpha} \\ \overset{\alpha}{Z}_{\gamma\alpha} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & + - 5.47\cdot 10^{-5} & -2.18\cdot 10^{-4} \\ +5.47\cdot 10^{-5} & 1 & +4.79\cdot 10^{-5} \\ +2.18\cdot 10^{-4} & -4.79\cdot 10^{-5} & 1 \end{bmatrix} \begin{bmatrix} -1300\ m \\ +300\ m \\ -200\ m \end{bmatrix}$$

500 meter local network, detailed computation

$$\overset{\beta}{X}_{\beta\gamma} = +800\ m - 83.9\ mm, \quad \overset{\gamma}{X}_{\gamma\alpha} = -1300\ m + 27.2\ mm$$

$$\overset{\beta}{Y}_{\beta\gamma} = -800\ m - 84.9\ mm, \quad \overset{\gamma}{Y}_{\gamma\alpha} = +300\ m - 80.7\ mm$$

$$\overset{\beta}{Z}_{\beta\gamma} = +150\ m - 5.8\ mm, \quad \overset{\gamma}{Z}_{\gamma\alpha} = -200\ m - 297.8\ mm$$

$$\overset{\alpha}{X}_{\alpha\beta} + \overset{\beta}{X}_{\beta\gamma} + \overset{\gamma}{X}_{\gamma\alpha} \neq 0: \quad -83.9\ mm + 27.2\ mm = -56.7\ mm$$

$$\overset{\alpha}{Y}_{\alpha\beta} + \overset{\beta}{Y}_{\beta\gamma} + \overset{\gamma}{Y}_{\gamma\alpha} \neq 0: \quad -84.9\ mm - 80.7\ mm = -165.6\ mm$$

$$\overset{\alpha}{Z}_{\alpha\beta} + \overset{\beta}{Z}_{\beta\gamma} + \overset{\gamma}{Z}_{\gamma\alpha} \neq 0: \quad -5.8\ mm - 297.8\ mm = -303.6\ mm$$

500 meter local network, misclosures

# BEISPIEL für ANHOLONOMITÄT

H. MORITZ: 3 OSU – Reports 1978

ROTATION – DREHIMPULS

H. MORITZ und I.I. MUELLER (1987): Earth rotation, UNGAR, New York

(i) Kinematische EULER-Gleichungen

$\rightarrow \omega = \mathrm{F}\text{-}1(\mathrm{d}\lambda, \mathrm{d}\theta, \mathrm{d}\psi)$

„EULER oder CARDAN Winkel"

$\leftarrow (\mathrm{d}\lambda, \mathrm{d}\theta, \mathrm{d}\psi) = \mathrm{F}\omega$

„FROBENIUS Matrix
von integrierenden Faktoren"

„ω Rotationskomponenten: CARTAN"

(ii) Dynamische EULER-Gleichungen

„gewichtete EULER-Gleichungen"

„Trägheitsmoment: symmetrische 3x3-Matrix"

# HEINRICH BRUNS

* 4. Sept. 1848 Berlin, gest. 23. Sept. 1919

MATHEMATIKER – ASTRONOM – GEODÄSIE

* Studium Mathematik, Astronomie und Physik in Berlin/Universität

LEHRER: KUMMER – WEIERSTRASS

1872 – 1973: Sternwarte PULMOWA

Bis 1878: Sternwarte DORPAT / Dozent an der Universität

1878: ausserord. Prof. für Mathematik: BERLIN

1882: **ord. Prof. für Astronomie/Sternwarte Leipzig**
**REKTOR**

in Berlin war er am Geodätische Institut tätig.

## Hauptaufgaben

- elliptische Integrale (DORPAT 1875)
- Figur der Erde (Berlin 1878)
- Astronomische Refraktion
- Integrale des Vielkörperproblems: 3 Körperproblem, ? Stabilität ? (1887: LEIPZIG)
- EIKONAL (1895)

  geometrische Optik / 1. FRESNEL-ZONE

## Potential-Theorie, GLEICHGEWICHTSFIGUREN

* BRUNSscher Polyeder
* Randwertaufgabe relativ zur Referenzkugel

**BRUNS → STOKES → VENING-MEINESZ**

VORTRÄGE von H. KAUTZLEBEN und E. BUSCHMANN

# „HÖHENSYSTEME“

(i) Dividiere das Potential/Potentialunterschiede durch eine Konstante, z.B. dem Mittelwert des Vermessungsgebietes

$$\frac{\mathrm{d}W}{\Gamma(\mathrm{mean})}$$

(ii) Dividiere das Potential/Potentialunterschiede durch die SOMIGLIANA-PIZZETTI Schwere in Reihen entwickelt (NANO - GAL)

„Ellipsoidisches Koordinaten-System“

$$\frac{\mathrm{d}W}{\Gamma(\mathrm{SOM-PI})}$$

A. ARDALAN und E: GRAFAREND

(NANO-GAL: J. GEODESY **75** (2001) 424-437)

# GRAVITOSTATICS

MASS POINTS
versus
EXTENDED BODIES

Isaac Newton
„Inertial reference frame“

„the gravitational field is conservative“
POTENTIAL
C.F. GAUSS – G. GREEN
„grad U“
INTERIOR VERSUS EXTERIOR
gravity field
LAPLACE EQUATION
div grad U = 0
„exterior“

multibody gravitation
Karl Heinz Ilk
A.M. FRIEDMAN and V.L. POLYACHENKO
Physics of Gravitation Systems I+II,
Springer, New York, 1984
tidal gravity field
CHANDRASEKHAR LEMMA
„The Center of the COSMOS moves inertially“
(constant speed)

multi-body Boundary Value problem

interior versus
exterior BVP: Krarup,
Grafarend - Sanso

Coupling of gravitation and rotation

Coupling of gravitation and rotation

forces:

(i) gravitation, (ii) centrifugal, (iii) EULER

„angular momentum“: length-of-day

polar motion ↔ precession-nutation

„NO POTENTIAL“

(iv) CORIOLIS: currents, flow

„NO POTENTIAL“

HELMHOLTZ decomposition

„grad - rot“

grad(U+V) + rot $\Gamma$

(i) div grad W(x) = $2\Omega^2$ („external“)

(ii) rot $\Gamma$ = $-2\dot{\Omega}$

# cont. GRAVITOSTATICS



Base functions

spherical harmonics
ellipsoidal harmonics

2-axis → 3-axis
„BIFURCATION“
**S. Chandrasekhar**

BJERHAMMAR-KRARUP
RUNGE-WALSH Approx. Theorem
ILL-POSED PROBLEMS

IMPROPERLY POSED
DOWNWARD CONTINUATION
BIAS-VERSUS
MEAN SQUARE ERRORS
REGULARISATION

BVP
fixed-free
Grafarend-Ardalan-Sideris
linear-nonlinear
1st, 2nd, 3rd, oblique
mixed

Reference Fields

GEOID-AREOID-PLANETS

TOPOGRAPHY

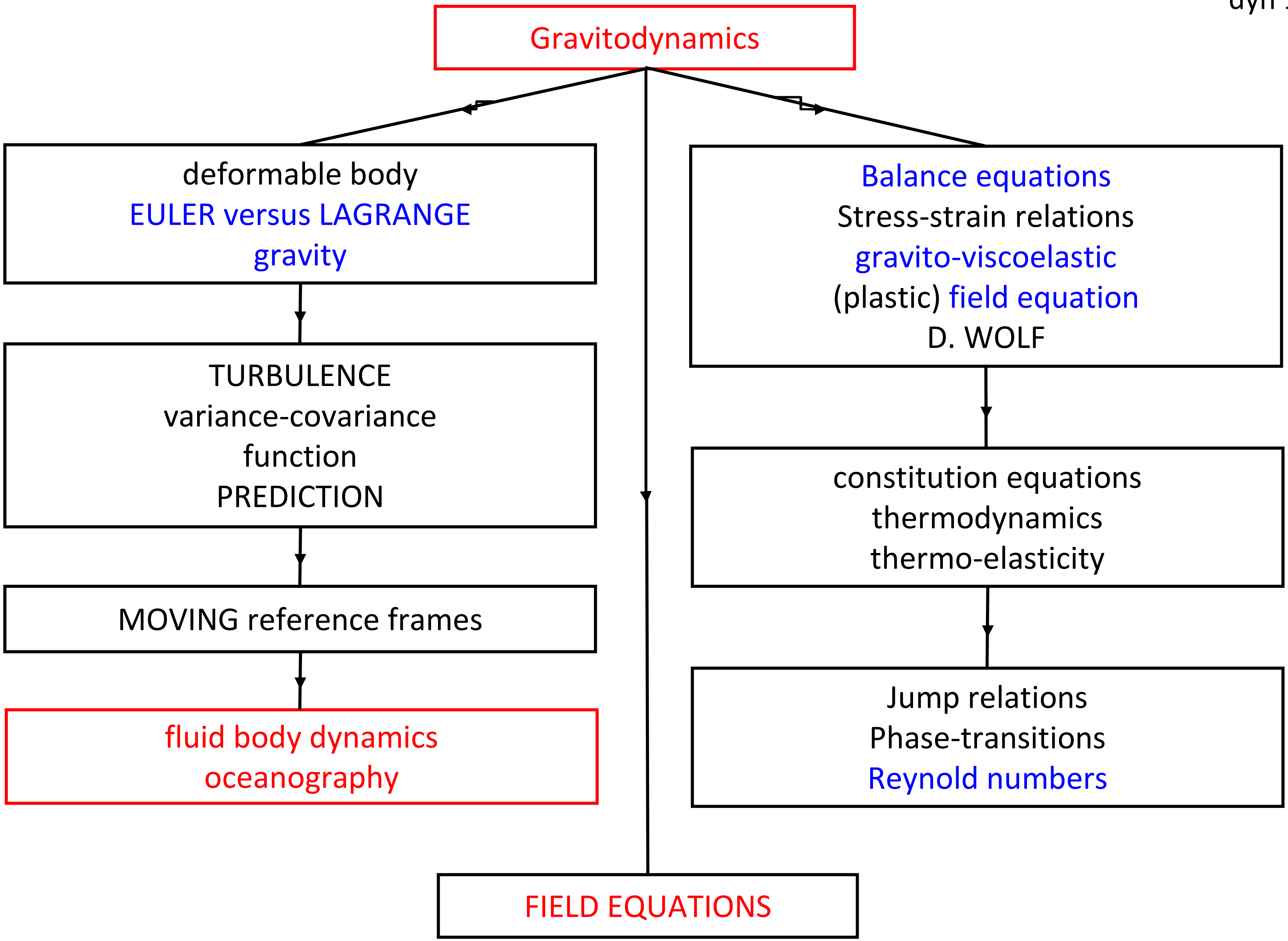
STOKES
BRUNS-VENING-MEINESZ
„generalisations“

PLANE-SPHERE-ELLIPSOID

SOMIGLIANA-PIZZETTI

forces
gravitation, volume forces, surface forces („stress“)
line forces (plasticity, dislocation: Earthquakes)

GRAVITODYNAMICS


Gravitodynamics
deformable body
EULER versus LAGRANGE
gravity
TURBULENCE
variance-covariance
function
PREDICTION
MOVING reference frames
fluid body dynamics
oceanography
Balance equations
Stress-strain relations
gravito-viscoelastic
(plastic) field equation
D. WOLF
constitution equations
thermodynamics
thermo-elasticity
Jump relations
Phase-transitions
Reynold numbers
FIELD EQUATIONS

# FIELD EQUATIONS

curl equ.:

$$\operatorname{rot}\boldsymbol{\Gamma} = -2\boldsymbol{\omega}\operatorname{div}\mathbf{x}^{\bullet} + \boldsymbol{\Omega}\operatorname{grad}\langle \mathbf{x}^{\bullet} \mid \boldsymbol{\omega}\rangle - -2\boldsymbol{\omega}\,\mathbf{x}\operatorname{rot}\mathbf{x}^{\bullet} - 2\boldsymbol{\omega}^{\bullet}$$

divergence equ.:

$$\operatorname{div}\boldsymbol{\Gamma} = 4\pi\rho + 2\omega^2 + 2\langle \boldsymbol{\omega} \mid \operatorname{rot}\mathbf{x}^{\bullet}\rangle$$

VORTICITY rot **x•**

rigid body

$$\operatorname{rot}\Gamma = -2\omega^{\bullet}$$

$$\operatorname{div}\Gamma = 4\pi g\rho + 2\omega^2$$

hydrostatic prepressed
elastic body

$$\boldsymbol{\Sigma} = -p\,\mathbf{I} + c\,\mathbf{E}$$

stress tensor

**balance equations**

$$\mathrm{d}_{tt}(\rho(\mathbf{x},t)) = \rho\mathbf{I} - \operatorname{grad} p =$$

$$= \rho\operatorname{grad}(\mathrm{U} + \tfrac{1}{2}|\boldsymbol{\omega}\times\mathbf{x}|^2) - \operatorname{grad} p$$

CHANDRASEKHAR'S VIRIAL METHOD

virial equations of order zero, one, two and higher order

EQUILIBRIUM figures of higher order

BIFURCATION

# cont. GRAVITODYNAMICS



EQUILIBRIUM FIGURES

Superpotential and
Supermatrix

homogeneous versus heterogeneous
Ellipsoids

THE VIRIAL THEOREM

THEORY OF BIFURCATIONS

? Stability ?

Maclaurin
JACOBI

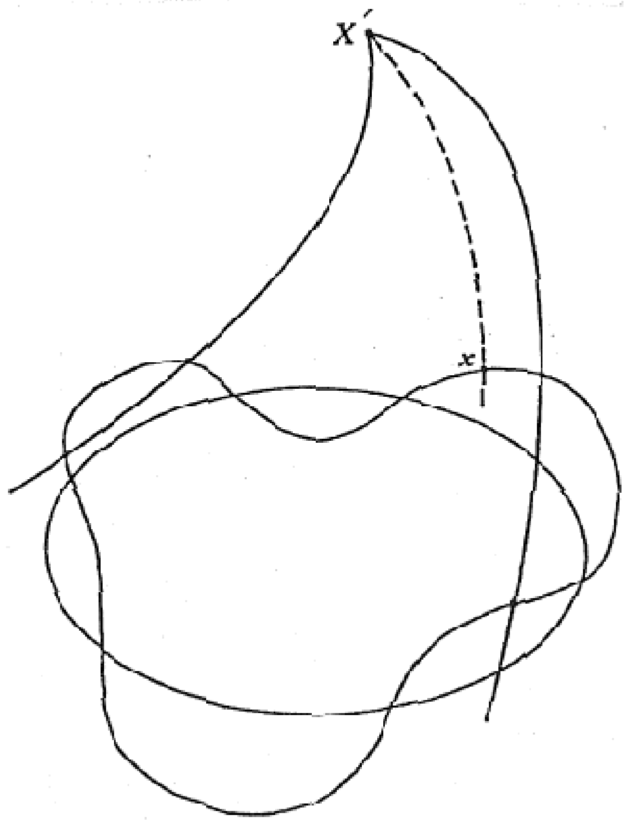
JEANS
ROCHE
DARWIN

DEDEKIND
RIEMANN

# Field lines of gravity, their curvature and torsion, the Lagrange and the Hamilton equations of the plumbline

The length of the gravitational field lines/of the orthogonal trajectories of a family of gravity equipotential surfaces/of the plumbline between a terrestrial topographic point and a point on a reference equipotential surface *like the geoid – also known as the orthometric height* – plays a central role in Satellite Geodesy as well as in Physical Geodesy. As soon as we determine the geometry of the Earth pointwise by means of a satellite GPS (Global Positioning System: «global problem solver») we are left with the problem of converting ellipsoidal heights (geometric heights) into orthometric heights (physical heights). For the computation of the plumbline we derive its three differential equations of first order as well as the three geodesic equations of second order. *The three differential equations of second order take the form of a Newton differential equation* when we introduce the parameter time via the Marussi gauge on a conformally flat three-dimensional Riemann manifold and the generalized force field, the gradient of the superpotential, namely the modulus of gravity squared and taken half. In particular, we compute curvature and torsion of the plumbline and prove their functional relationship to the second and third derivatives of the gravity potential. For a spherically symmetric gravity field, *curvature and torsion of the plumbline are zero*, the plumbline is straight. Finally we derive the three Lagrangean as well as the six Hamiltonian differential equations of the plumbline, in particular in their star form with respect to Marussi gauge.

Erik W. Grafarend (1997): Annali di Geofisica **40** (1997) 1233 - 1247




*Projective heights in gravity space*, minimal distance mapping with respect to a reference equipotential surface, the *geoid* $w_0 = (62, 636, 856.5 \pm 3)\ \mathrm{ms}^{-2}$, *at a reference epoch* $t_0$, *orthometric height* $h_G$ (length of the plumbline / orthogonal trajectory from $X \in T^2$ to $x \in$ GEOID), representation of the *geoid* with respect to an ellipsoid of revolution $E^2_{a,b}$ according to J. ENGELS and E. GRAFAREND (1992a, b)

Duality between horizontal and vertical fields in $\{\mathbb{R}^3, \delta_{ij}\}$ equipped with a Euclidean metric $\delta_{ij}$.

*Normalized tangent vector of the plumbline is identical to the surface normal vector of an equipotential surface*

1st order differential equations

$$\frac{dx}{dS} = -\operatorname{grad} w / \| \operatorname{grad} w \| \sim \frac{dx^k}{dS} = -\partial_k w / \sqrt{\delta^{lm} \partial_l w \partial_m w}$$

2nd order differential equations

$$\frac{d^2x^k}{dS^2} + \frac{\gamma, l}{\gamma^3}(\gamma^2\delta^{kl} - \gamma^k\gamma^l) = 0$$

$$\boxed{\text{Marussi gauge}}$$

$$\| x^{\bullet} \| = \| \operatorname{grad} w \|, \quad \frac{dx}{dS} = \| x^{\bullet} \|^{-1} x^{\bullet}$$

1st order differential equation of the plumbline in Marussi gauge

$$\boxed{x^{\bullet} = -\operatorname{grad} w}$$

2nd order differential equation of the plumbline in Marussi gauge

$$x^{\bullet\bullet k} = -(\partial_l \gamma^k) x^{\bullet l} = (\partial_l \gamma^k) \partial_l w = (\partial_l \gamma^k) \gamma^l$$

$$\boxed{x^{\bullet\bullet k} - \frac{1}{2} \partial_k \gamma^2 (x^m) = 0}$$

# Molodensky potential telluroid based on a minimum-distance map. Case study: the quasi-geoid of East Germany in the World Geodetic Datum 2000

The telluroid as introduced by M.S. Molodensky (1945, 1948, 1960) may be considered the best analytical representation of the irregular surface of the Earth. Given the placement vector of a point in geometry space, for instance by GPS (global problem solver), and reference gravity potential in gravity space, the telluroid can be uniquely defined by a properly chosen projection. For instance, astronomical longitude/astronomical latitude (spherical coordinates in gravity space) at a topographic point could be defined to coincide with reference longitude/reference latitude (spherical coordinates in reference gravity space) at a telluroid point in order to establish an isoparametric mapping of the

telluroid. Bode and Grafarend (1982) extensively studied such an isoparametric telluroid mapping with respect to a reference gravity potential which is additively decomposed into (1) the zero-order coefficient of a spherical harmonic expansion of the gravitational potential and (2) the centrifugal potential. In particular, they succeeded in identifying the singular points of such a telluroid map. Here we aim at an ellipsoidal telluroid mapping, which is set up as follows.

A. A. Ardalan, E.W. Grafarend, J. Ihde
Journal of Geodesy (2002) 76: 127-138

# Best Fit, Som – Pi Field

Analytically we can formulate the above-stated optimisation problem by minimising the constraint Lagrangean

$$\begin{aligned}\mathbf{L}(x_1,x_2,x_3,x_4) &:= \|\mathbf{x}-\mathbf{X}\|^2 + x_4(W_P - w_p)\\ &= [x - X(x_1,x_2,x_3)]^2 + [y - Y(x_1,x_2,x_3)]^2\\ &\quad + [z - Z(x_1,x_2,x_3)]^2\\ &\quad + x_4[W(x_1,x_2,x_3) - w_p]\\ &= \min_{x_1,x_2,x_3,x_4}\end{aligned} \tag{1}$$

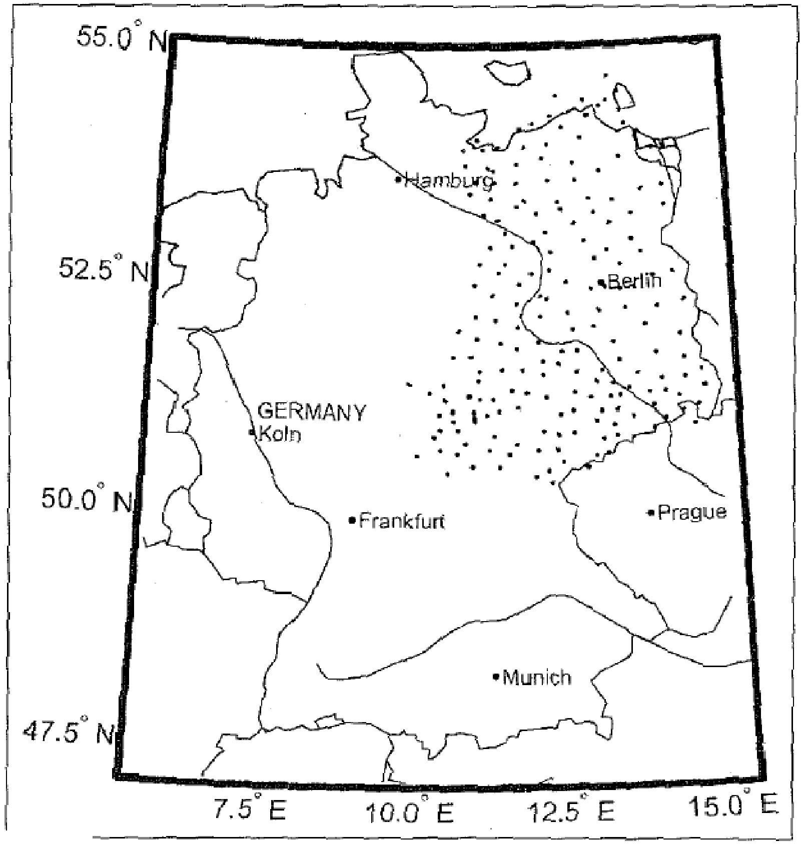
$$(\hat{x}_1,\hat{x}_2,\hat{x}_3,\hat{x}_4) = \min_{x_1,x_2,x_3,x_4} \mathbf{L}(x_1,x_2,x_3,x_4) \tag{2}$$

**Definition ~~1~~.** *Somigliana–Pizzetti field as the gravity field of a rotational ellipsoid.*

Explicit form in terms of fundamental geodetic parameters $\{a, b, W_0, \Omega\}$ (according to Grafarend and Ardalan 1999)

$$\begin{aligned}W(\phi,u) = {}& \left(W_0 - \frac{1}{3}\Omega^2 a^2\right)\frac{\cot^{-1}(\frac{u}{\varepsilon})}{\cot^{-1}(\frac{b}{\varepsilon})} + \frac{1}{3}\Omega^2(u^2+\varepsilon^2)\\ &+ \left\{\frac{1}{3\sqrt{5}}\Omega^2 a^2 \frac{(3\frac{u^2}{\varepsilon^2}+1)\cot^{-1}(\frac{u}{\varepsilon}) - 3\frac{u}{\varepsilon}}{(3\frac{b^2}{\varepsilon^2}+1)\cot^{-1}(\frac{b}{\varepsilon}) - 3\frac{b}{\varepsilon}}\right.\\ &\left. - \frac{1}{3\sqrt{5}}\Omega^2(u^2+\varepsilon^2)\right\}\frac{\sqrt{5}}{2}(3\sin^2\phi - 1)\end{aligned}$$

# Case study: potential quasi-geoid of East Germany




Figure

**Fig. 2.** The 196 GPS stations in the eastern part of Germany. Equidistant conic projection; standard parallels: 50°N and 52.5°N; reference ellipsoid: WGD2000

# Molodensky potential telluroid

**Table 2.** Part of the GPS file of the eastern part of Germany

| Longitude ($l_p$) [deg] | Latitude ($b_p$) [deg] | Ellipsoidal height ($h_p$) [m] | Geopotential number [$m^2/s^2$] |
|---|---|---|---|
| 13.4363 | 54.6772 | 82.295 | 455.75 |
| 13.6433 | 54.5136 | 68.192 | 318.76 |
| 12.5016 | 54.4716 | 39.663 | 21.20 |
| 13.0076 | 54.4256 | 48.669 | 115.93 |
| 13.4252 | 54.4172 | 92.697 | 553.89 |
| 13.2909 | 54.3508 | 42.444 | 58.39 |
| 12.7371 | 54.2981 | 50.352 | 127.90 |
| 13.6586 | 54.2971 | 101.363 | 641.58 |
| 13.0796 | 54.2511 | 55.762 | 185.67 |
| 12.4053 | 54.2495 | 40.304 | 23.93 |

**Table 3.** Transferred Jacobi ellipsoidal coordinates $\{\lambda, \phi, u\}_p$ of the Gauss ellipsoidal coordinates given in Table 2

| $\lambda_p$ | $\phi_p$ | $u_p$ |
|---|---|---|
| 13.4363 | 54.5864 | 6 356 834.2477 |
| 13.6433 | 54.4226 | 6 356 820.1294 |
| 12.5016 | 54.3805 | 6 356 791.5681 |
| 13.0076 | 54.3345 | 6 356 800.5844 |
| 13.4252 | 54.3261 | 6 356 844.6627 |
| 13.2909 | 54.2596 | 6 356 794.3526 |
| 12.7371 | 54.2069 | 6 356 802.2698 |
| 13.6586 | 54.2059 | 6 356 853.3393 |
| 13.0796 | 54.1598 | 6 356 807.6861 |
| 12.4053 | 54.1582 | 6 356 792.2103 |

# Telluroid Map: East Germany

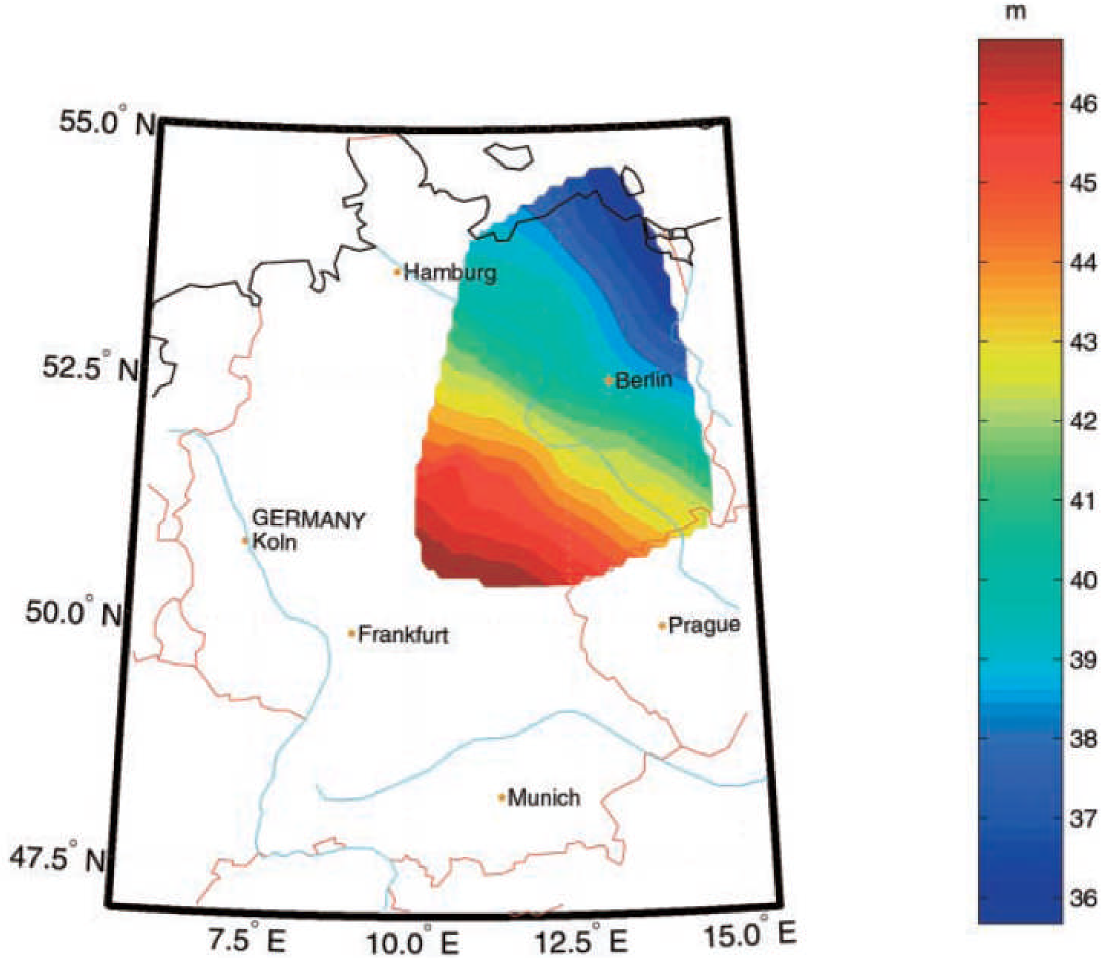



**Figure** Quasi-geoid map of East Germany, based on the minimum-distance mapping of the physical surface of the Earth to the Somiglian Pizzetti telluroid. The quassi-geoid undulations are in the interval 35.609 to 47.501 m. Equidistant conic projection; standard parallels:50°N and 52.5°N; reference ellipsoid: WGD2000