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Dynamical Systems with Complex Hysteretic Nonlinearity

Introduction

Hysteresis is a well-known phenomenon in many branches of science [Bertotti06]. It refers to situations, where for a given external parameter multiple internal system states are possible. Which one of these states is assumed, depends on previous parameter variations and therefore on the history of the system. Classical examples are magnetic materials, where magnetization and external magnetic field are hysteretically related. Sometimes one encounters bistable situations corresponding to only one single hysteresis loop describing the input-output or field vs. state relation. We are, however, interested in complex hysteretic systems with arbitrary many internal states, which correspond to a given value of the external field or input. Correspondingly, these ubiquitous systems in addition to a major hysteresis loop, show sub-loops, sub-sub-loops, etc. as the input is varied. Apart from magnetic materials such behaviour is found e.g. in piezo-electric materials, shape memory alloys, superconducting systems, porous materials such as soils, and also in economic systems. While the hysteretic behaviour of these systems by itself has been investigated quite extensively, not much is known about scenarios that may arise if hysteretic subsystems are coupled dynamically to its environment. This amounts to consider dynamical systems with hysteretic nonlinearity.

Models

We treat this problem by adopting a well-known model for the hysteretic subsystem, the so-called Preisach model [Mayergoyz91]. It is defined by the following operator, which maps an input time series $x(t)$ to an output $y(t)$ by superimposing with weights $\mu(\alpha,\beta)$ the outputs $s_{\alpha\beta}$ of infinitely many

$$y(t) = \mathcal{P}\left[x(t)\right] = \iint d\alpha d\beta\, \mu(\alpha,\beta)\, s_{\alpha\beta}\left[x(t)\right]$$

elementary hysteresis loops

The elementary hysteresis loops are rectangular loops with outputs +1 or -1 and switching values α (upper) and β (lower) (delayed relay or Schmitt trigger). Formally the output is given by

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$$s_{\alpha\beta}[x(t)] = \begin{cases} +1 \text{ if there exists } t_1 \in [t_0, t] \text{ s.t. } x(t_1) \geq \alpha \text{ and } x(\tau) > \beta \text{ for all } \tau \in [t_1, t] \\ -1 \text{ if there exists } t_1 \in [t_0, t] \text{ s.t. } x(t_1) \leq \beta \text{ and } x(\tau) < \alpha \text{ for all } \tau \in [t_1, t] \\ s_0 \in \{-1, +1\} \text{ if } \beta < x(\tau) < \alpha \text{ for all } \tau \in [t_0, t] \end{cases}$$

The weight function $\mu(\alpha,\beta$ is typically taken as a 2-dimensional Gaussian restricted to the region $\beta< \alpha$ in the so-called Preisach plane, or a constant in the triangle $-1<\beta< \alpha<1$. In the following we assume the latter. The Preisach model is known to have universal properties and has been used by physicists and engineers in modelling all of the above mentioned systems [Mayergoyz91, Bertotti06]. An important property of the Preisach model is that it can store certain extremal values of the previous input time series by remembering which of the elementary relays have switched and which not. In this way it can develop memory, which in principle can become infinitely long. This memory is typically fluctuating in time with the fluctuations depending on the input time series in a non-trivial way. The currently stored memory sequence determines in addition to the current input the output of the Preisach model.

We couple this model with ordinary dynamical systems in the following way. In model A and B we use the Preisach model simply as a transducer: We generate the input for the Preisach model from deterministic or stochastic dynamical systems and observe the output of the Preisach model. We use for model A as input time series those generated by the logistic map,

$$x(t+1) = f(x(t)) = 1 - a\,(x(t))^2$$

whereas for model B we generate the input by the stochastic Ornstein-Uhlenbeck process

$$\frac{d}{dt}x(t) = -\frac{1}{\tau}\,x(t) + \xi(t)$$

where $\xi(t)$ denotes Gaussian white noise with zero mean and unit variance. τ is the correlation time for this process. We will see that despite the simplicity of these input processes, the output shows quite complicated behaviour. Model C provides a simple example for a nonlinear hysteretic feedback system: We apply to the output of the Preisach model the nonlinear logistic map, the result is used as new input for the Preisach model, the output of the latter is again iterated by the logistic map, then fed into the Preisach model, etc. Thus we repeatedly iterate the *logistic map with Preisach nonlinearity* consisting of one application of the Preisach transducer followed by one

$$x(t+1) = \mathcal{P} \circ f(x(t))$$

application of the logistic map
Note that the action of P depends in principle on all previous input values.

Results

The complexity of the output of a hysteretic system a described by the Preisach model when subject to a chaotic input (Model A) is demonstrated by the following results. We use as input the fully chaotic time series generated by the logistic map for the parameter value *a*=2. On the left of Fig.1

we show for reference the return map for the input $x(t+1)$ vs. $x(t)$, which is just the logistic parabola, and the subject of interest, the return map $y(t+1)$ vs. $y(t)$ of the corresponding output of the Preisach model. The many branches in the latter reflect the high-dimensionality of the system. Although details of this structure are not yet fully understood,

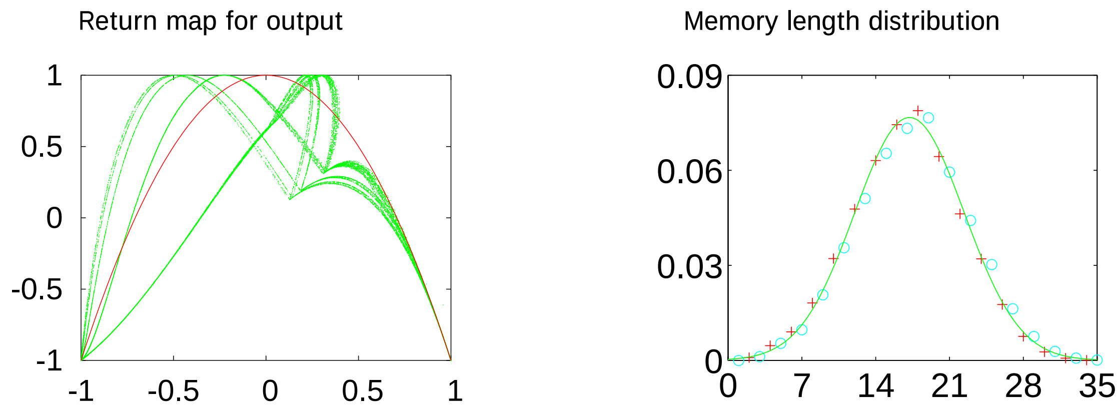

Fig.1: *The return map (left) for the output of the Preisach model (green) is shown in comparison to the return map of the chaotic input, the logistic parabola (red). The structure in the output return map is attributed to the fluctuating memory of the Preisach model with its memory length distribution shown in the right panel.*

we know that the branches can be classified according to the parity (odd or even) of the memory length. The memory length is defined as the number of values needed to specify in the Preisach plane ($\alpha-\beta$ plane) the piecewise constant separation line, which separates elementary relays being in state +1 and -1, respectively. Since this line, and correspondingly the system memory and its length, changes dynamically depending on the input, we deal here with a class of dynamical systems which are currently not well understood: The memory length can be regarded as an effective dimensionality of the dynamical system, and therefore this is an example of a system with a time-dependent effective dimension (the "true" system dimension is infinite). The distribution of this effective dimension or memory length is shown on the right of Fig.1. There is a slight difference in the distribution of odd (green circles) and even values (red crosses), but as the green line shows, they are approximately Gauss distributed with mean value around d=18. Since the properties of the memory are input dependent one finds a quite different behaviour e.g. for the fractal input generated for the parameter a corresponding to the Feigenbaum accumulation point of the logistic map. More details on this system and the behaviour of the memory can be found in [Schubert05].

Instead of a deterministic chaotic input one may be interested also in the system behaviour under stochastic input (Model B). Some aspects of this problem were treated for the Ornstein-Uhlenbeck (OU) process as input and for the special case of symmetric hysteresis loops in [Dimian04]. Here we are interested in the dependence of the output properties as the correlation time τ of the input is varied. In Fig.2 the spectral density of the output is shown for three different values of τ. While the input spectrum is always a Lorentzian, the output spectrum and correspondingly the structure of its autocorrelation function is more complicated. One aspect is that the width of the spectrum becomes very narrow for small input correlation times.

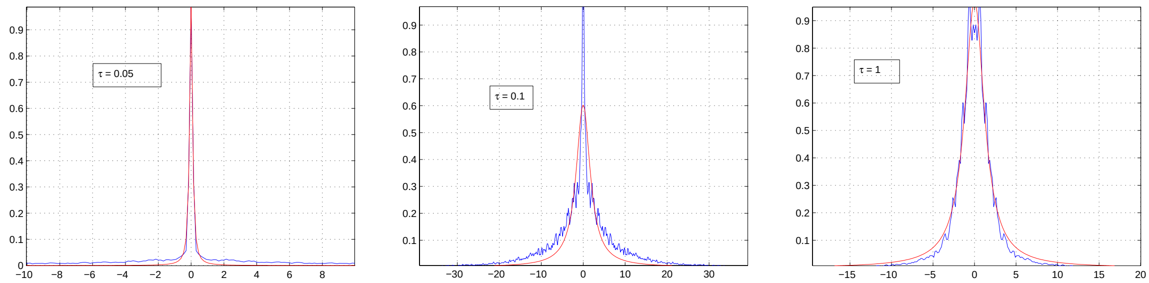

Fig.2: *The spectral density of the output is shown for three OU input signals, which differ only in its correlation time τ. The red line is a fit by a Lorentz profile.*

Actually it seems to diverge for τ->0, a result which is supported also by some recent analytic results. For increasing τ the spectrum develops more structure with clear deviations from a Lorentz profile. These deviations become even stronger for systems with a Gaussian Preisach density $\mu(\alpha,\tilde{\beta}$. More details also on the related variation of the mean memory length with τ can be found in [Heße05]. The origin of these structures, however, is not yet clear and needs further investigations.

The following results for model C seem to be the most complicated to understand. Some aspects of dynamical systems with Preisach nonlinearity have been published recently for a driven iron pendulum [Donnagain06]. In contrast to that our discrete time hysteretic dynamical system is much simpler and allows e.g. for a systematic investigation of the dependence on initial conditions. In Fig.3 we show the asymptotic state of the system as the initial value x_0 is changed (as initial state of the Preisach model all elementary relays are set to -1, the parameter of the logistic map is a=2). The asymptotic state may be simply a fixed point or a chaotic or other complicated orbit. The lines in Fig.3 (e.g. for $x_0 < -0.5$) correspond to lines of fixed points, i.e. the attracting fixed point varies continuously as the initial condition is changed: The system exhibits infinitely many attractors, in this case point attractors. One observes, however, also more complicated asymptotic objects, which are visible as scattered points in the x*- direction. Probably these are chaotic objects, but this needs to be investigated in more detail, too. Another interesting aspect of Fig.3 is the apparent approximate self-similarity of the asymptotic structures in dependence of the initial conditions. E.g. the region of initial conditions between 0.2454 and 0.25 (from cusp to cusp on the right of Fig.3) seems to be similar to the region from -0.187 to 0.675 (left of Fig.3).

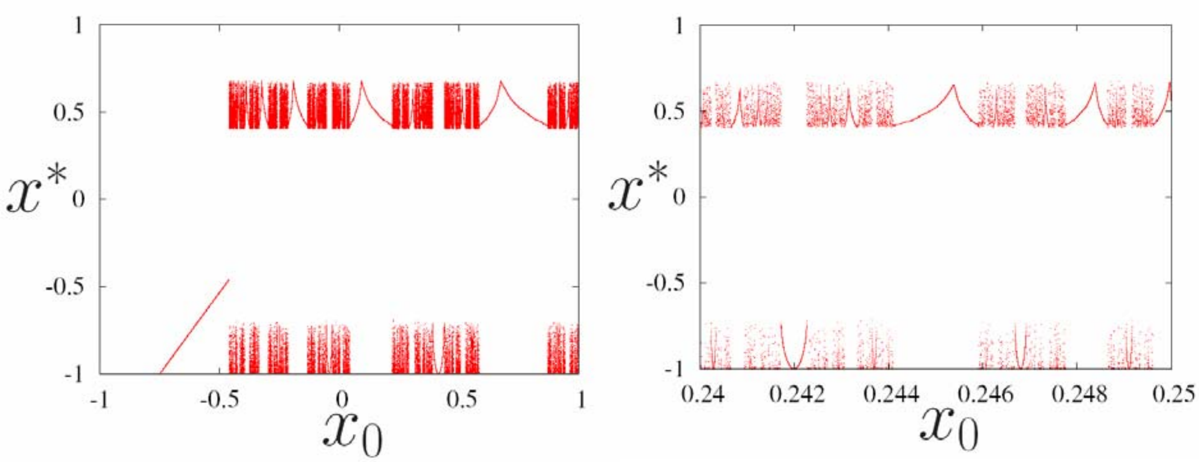

Fig.3: *The dependence of the asymptotic state of model C in dependence of the initial condition is shown. On the right we show an enlargement of the region between 0.24 and 0.25 of the left figure indicating a self-similar structure.*

For other parameters of the logistic map the asymptotic state of the map with Preisach nonlinearity shows also similar structures. In general the dissipative nature of the hysteretic Preisach transducer tends to suppress chaotic motion. For fixed initial condition x_0 one finds complicated bifurcation scenarios as the parameter *a* is varied. More results can be found in [Lange05]. Some of them, e.g. the line of fixed points for $x_0 < -0.5$ are easily understood. A better understanding of the self-similar structures of Fig.3 and the mentioned bifurcation scenarios, however, needs further investigations.

Summary

We have demonstrated that the coupling of simple dynamical systems with a typical hysteresis model leads to interesting new scenarios in nonlinear dynamics. This is a result of the ability of hysteretic systems to store information about previous input dynamically. The time-dependent memory, or fluctuating dimension of the system provides on one hand a challenge for the analysis of such systems. On the other hand such systems are of great practical importance and therefore are a rewarding subject of future research in nonlinear dynamics.

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