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# On inverse problems studied by Helmut Moritz and on his human activities

## A) About stability of inverse problems

Abstract: The paper emphasizes Moritz's philosophical point of view, i. e. fundamental concepts concerning the inverse problems: *definitions*, *kinds*, *properties*, *indetermination problems* and *solution methods*. Further on, the universality of theory of inverse problems is presented through its application in various fields, first of all in Geodesy, then in Geophysics, Technology and Medicine. In this way the importance of Moritz's wide-scope and, certainly, deep views of these problems is emphasized. The paper points out the *main causes of ill stability* of the mathematical models by which we describe inverse problems. Also, *recommendations* have been made *what to do for the improvement of the model stability*.

## Introduction

Any law of nature or, in some cases, of social development, can be expressed as a description of the character or structure of mutual relations existing among the considered *phenomena* (*systems*, *processes*, *objects*) called *variable quantities* or, briefly, *variables* (Perović, 2005, Aivazyan et al., 1985). Any relation of this kind can be described by a *mathematical model*. For example, Newton's laws of mechanics and, based on them, all classical mechanics form a set of models of mechanical phenomena. Maxwel's equations in physics give rise to mathematical models of electro-dynamical phenomena, etc.

However, there are also such phenomena for which, in principle, it is impossible to construct a model in a purely mathematical way. As an example we have a human being or a human collective performing some functions, an institute, economy branch, economics and so on. However, the actual progress of science and technology requires the formation of mathematical models for such systems as well (Perović, 2005).

In general, we wish to describe every considered phenomenon (system, process, object) by means of an adequate mathematical model, linear or nonlinear. This is because it is the best way to make an analysis of the phenomenon, and likewise the presentation of the results achieved.

It is to such tasks that inverse problems belong.

Moritz particularly discusses inverse problems, primarily in Geodesy (Moritz, 1989, 1990, 1993a, 1993b), but also, in Geophysics, Technology and Medicine (Moritz, 1993b).

Tikhonov and Arsenin (1977) provided the methods for solving Ill-Posed Problems. Anger (1993) also considers the Basic Principles of Inverse Problems.

Perovic investigated inverse problems in the following fields: *in geodetic networks* (Perović, 2001, Perović et al. 2006b), *aircraft laser scanning* (Perović, 2006a, 2007), and geodetic measurements – in which he examined *variance components* of angles, distances, height differences (geometric and trigonometric levelling) and GPS positioning (Perović, 2007, 2008; 2005, Chs. 21 and 32). Besides, he dealt with *mathematical model stability*, especially linear (Perović, 2005; Chs.: 2, 6, 7, 8, 11, 21, 23-29, 32, 34, 36-39).

An exhaustive review of the papers about inverse problems in different fields can be found in Anger et al. (1993).

## The main points of inverse problems

In his introductory paper „*General Considerations Regarding Inverse and Related Problems*" Moritz (1993b) discusses all of these problems, expounding his philosophical view, in mathematical sense divides them into *direct* and *inverse* problems, and for inverse problems states the fundamental concepts: *definitions, kinds, properties, indetermination problems* and *solution methods*. This classification into direct and inverse problems is substantial because solutions of direct problems are uniquely defined, while *for inverse* *problems there may be infinitely many solutions or no solutions at all*. Other differences between these two problems will be discussed furter on.

For a mathematically properly posed problem to correspond to reality the following basic conditions (Hadamard's (1902) conditions) must be satisfied (Moritz, 1993b):

1. *Existence* (The solution must exist);
2. *Uniqueness* (The solution must be uniquely determined by the data), and
3. *Stability* (The solution must depend continuously on the data).

These conditions satisfy direct problems.

If one or more of these requirements are violated, then we have *improperly posed*, or *ill-posed problems*. In fact, *most inverse problems are improperly posed*.

Consider the following symbolic expression

$$g = Af \quad , \tag{1}$$

where $A$ denotes *operator* which acts on $f$ and produces $g$. Mathematically, $g$ is a (generalised) *function* of $g$. Operator $A$ may be linear or nonlinear.

The *direct problem* is described by equation (1) and we shall assume that *the operator A* of the *direct problem is stable*, i. e., that „*small causes have small effects*".

Solving (1) formally by

$$f = A^{-1}g, \tag{2}$$

where $A^{-1}$ denotes *inverse operator* which acts on $g$ and produces $f$, we get an *inverse problem*. According to Moritz (1993b), we have the following *classification*:

| Given | To be Determined | Kind of Problem |
|---|---|---|
| $A$, $f$ | $g$ | Direct Problem, |
| $A$, $g$ | $f$ | Inverse Problem of First Kind, and |
| $f$, $g$ | $A$ | Inverse Problem of Second Kind. |

Moritz (1993b) also adduces examples from different fields:

- *For direct problems*: a linear transformation, the Newtonian potential, a computer program, projection, the evaluation operator, classical mechanics and deterministic chaos;
- *For inverse problems of first kind*: linear vector transformation, gravimetric inverse problem, interpolation, photogrammetry, tomography and human perception, and
- *For inverse problems of second kind*: determination of transformation matrix, symmetric matrix, rotation matrix, system parameters (physical, technological etc), determination of physical law, statistical test of hypotheses, diagnostics in medicine, induction: verification and falsification of hypotheses and boundary value problem of geodesy.

Besides these, there are many other tasks of inverse problems. One can mention the following: variance component estimation of measurements in order to define the measurements method precision, the determination variances of particular variables in order to obtain given variance of function of these variables, optimisation of geodetic networks, the determination of the orbital elements of binaries, etc.

Moritz (1993b) states that inverse problems of first kind are overdetermined, or both – overdetermined and underdetermined problems. The overdetermined problem is ill-posed, but, the introduction of "least squares" makes the problem well-posed.

Generally, inverse problems don't satisfy one or more Hadamard's conditions concerning properly posed problems, so that we have an *improperly posed*, or *ill-posed problems*; although there are properly posed problems, for instance *rotation matrix*.

Scientists use the terms "*improperly posed*" or „*ill-posed*". These terms mean that the considered process *is not described by adequate* (*mathematical*) *model*. Why does an inadequate model occur? For one thing because in most cases *we don't know the structure* (*system*) *parameters of the considered process*; thus they are unknown to us because we have not the „key" by which we could „unlock and peer into the interior of the process" and in that way we could see, i. e. we would be informed on, the process structure. After that we could describe the process by an adequate mathematical model. However, there also exist other causes of ill condition of system (see the next Section). Also, given the technological progress, new possibilities occur every day.

In addition to ill condition, mathematical and numerical solutions to inverse problems play an important role. However, they have been well studied and will not be dealt with here, we shall point out only the fact that the present author has found out several *new methods of solving non-linear problems*, published in the monograph Least Squares (Perović, 2005). The methods are as follows: *PERG non-linear LS* (Sec. 38.3), *The PERG Singular Non-linear LD Method* (Subsec. 38.4.1) and *The PERG Singular Non-linear LS Method* (Subsec. 38.4.2).

**The main causes of unsatisfactory solutions**

We said that inverse problems are *ill stable*, which is the principle cause unsatisfactory solutions. But we are also infrequently in a situation that the mathematical model is adequate but the solution is unsatisfactory. The cause of this is the *ill stability* of the mathematical model that may occur out of a number of reasons. The reasons are as follows.

1) *Some parameters of the system are almost co-linear* in the mathematical model. In that case there is nearly functional dependence of the corresponding parameters of the model. For example, if the orbit period of the binary is 200 years, and we conducted these measurements at the time interval of 20 years after every fifth year (there are 5 measurement points), then this will be the ill-posed linear system (after linearising), although the model is quite adequate (the orbit is eliptical one), since we made measurements only at 1/10 of the orbit. The condition number of the matrix of the normal equations is of the order of $10^3$-$10^4$, so that the propagation of errors is large, and the error estimate of the large ellipse axis will be of the order of 1-5″.
2) *The large measurement errors*. In that case system parameter estimates can deviate much from the exact values. An example. Suppose the major axis of the binary orbit of $a$ is 0.5″, while the precision of the speckle interferometry observations is of order of 0.01″ (Scardia et al.), that is, of the order of $10^{-2}$ with regard to the major orbit axis. Then, even under the condition of a good geometrical stability of the model the errors of the estimated parameters will be of the order of $10^{-2}$, and larger.
3) *A small number of measurements*. This will not lead to the satisfactory accuracy of system parameter estimates, not even by *the application of the asimptotic sampling theory*. Thus, in many applications of tomography to the problems of science and industry the number of data is small, say 20 – 200 (Natterer, 1993), in which case the system parameter estimates will not be reliable nor accurate enough.
4) *The irregular systematic errors in measurements*. Then:

4a) *To decrease the influence of the irregular systematic errors it is necessary to significantly increase the measurement interval*. Thus, when examining the accuracy of the relative GPS positioning, on the basis of 72 hour measurements of the basic vector „Kragujevac-Batočina" (Serbia) 20.7 km in length, for the coordinate difference according to $\varphi$ the present author got random errors standard $\sigma_{\varepsilon,\varphi} = 8.0\,\text{mm}$, and systematic – variable in time (Tropo + Iono) $\sigma_{\delta,\varphi} = 6.7\,\text{mm}$, where the variable errors are randomized only in the large measurement interval $T$ – larger than 2 hours; accordingly, $(T \geq 2)$, which yields:

| Measuring Interval, $T$ : | 0 h | 2 h | 8 h | 24 h |
|---|---|---|---|---|
| $\sigma_{\delta,\varphi(T)}$, mm: | 6.7 | 3.3 | 1.7 | 1.0 , |

while the random errors after 0.5 hour of measuring reduce under 1 mm;
4b) The application of asymptotic sampling theory yields better results if there are no these errors. Therefore *the irregular errors should be eliminated or their influence should be reduced*;
5) *The presence of the contaminating distributions* (as a result of larger random errors). These influences can be successfully reduced by means of the application of *robust methods*.

**Recommendations for the improvement of the model stability**

To have well posed and stable inverse problems the present author recommends the following:
I. At first, *the structure of an inverse problem should be well studied*, i. e. *adequate functional relationships between phenomena parameters should be established*, and after that any other theory is to be applied. This is usually a long and painstaking task, quite often an expensive one, too;
II. To *optimise the geometry* of the mathematical model and consequently *to make a better model stability*. We have to study the way of constructing of linear or non-linear model and find out the optimal, or at least acceptable, model geometry;
III. *To apply robust methods* for the elimination or for reducing the effect of gross errors.
IIIa) One of these methods is „*Robust PEROBLS 3 Method of Least Squares*" (Perović, 2005, Ch. 21, Sec. 21.5). The properties of the Robust PEROBLS 3

Method are: 1) Unbiasedness of estimates for model parameters, for corrections and for variance coefficient, 2) Correct degrees of freedom number, 3) Applicability for both known and unknown variance coefficient, 4) Stability of estimates, 5) The cofactor matrices of parameters and corrections estimates are calculated according to exact formulae, and 6) The hypotheses testing can be also done following the standard LS procedure, without any modifications;
IIIb) For *the estimation of variances of both distributions*, the basic and the contaminating one, for measurements of the same precision good results are yielded by the method „*Robust PEROBVC ML Method of Variance Estimating in the Tukeyan Mixed Distributions of Empirical Variances*", (Perović, 2005; Ch. 32, Sec. 32.14);
IV. Specially, *the irregular systematic errors in measurements should be well studied*, *and*, *also*, *the ways of reducing its influence in measurements*; for example by increasing the measuring time interval, and
V. Significantly *to increase a number of measurements*, in order to detect gross errors more reliably and in order to apply *the asimptotic sampling theory*.

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## B) On Humane Activities of Helmut Moritz

Abstract: Helmut‘s absolute readiness to help the people in need of assistance is hereby described. He has not only helped those who ask him for this, but has offered assistance immediately to those he finds to be in need of assistance. He is described as a person with absolutely positive energy; more precisely as a *person with no negative energy*. *He is exactly the person preached about in the Holy Scripture* !

When a scientist has mastered beautifully in fundamental area of his sphere, like for instance inverse problems, then he is able to comprehend the universality of that theory and to apply it to unrelated fields, just as Helmut Moritz has shown that the theory of inverse problems is applicable in Geodesy as well as in Geophysics, Technology and Medicine. It is one of sure – moreover sufficient, signs that tell of a great scientist, such as Gauss, Helmert, Krasovski and Tesla, to mention just a few. Such scientists are also great humanists.

We can say quite a lot about Helmut’s humanity and his character’s features. But, one should be a great writer, such as F. Dostoyevsky, to be able to write something about the person of Helmut Moritz. But, since I am not such a person, I will try on the basis of our constructive personal contacts and frequent long-time correspondence and telephone conversations, to present only a few of my observations on Helmut's humane person.

*Those who seek his assistance get it at once.* As a guest of the Institute for Geodesy in Belgrade in 1986, Helmut was asked to review a doctoral dissertation of a colleague. He read the dissertation during the following night (Helmut can speak and has the knowledge of the Serbian language) and the next morning he gave to the above-mentioned colleague some suggestions and remarks. On my request he immediately accepted to be a reviewer of my monograph Least Squares, for both texts – in English and in Serbian. During the bombing of Serbia and Belgrade in 1999 he accepted without thinking to assist a student of the post-graduate studies at the Electrical engineering in Belgrade with regard to the doctoral study in Vienna (previously the company SIEMENS where she is employed moved to Vienna); just imagine how long I had waited for his positive answer – less than two hours; assistance was, of course, warmheartedly given. And not only once!

*He gives his assistance kindly*, *as soon as he sees that one is in need of help*. Prior to the Conference of IAG in Birmingem in 1999, Helmut asked me over the phone: „Gligorije, what do you suggest to me to do about Yugoslavia (she could have been dismissed from membership due to non-paying of the membership fee)“. I gave him a proposal, and on the second day of the Conference I received a fax from him: „Gligorije, I have made it, Yugoslavia has not been ejected from the membership, and the category of the non – paying members has been introduced, so that Yugoslavia has been included into that category“. In the course of bombing of Serbia and Belgrade in 1999 Helmut encouraged us almost every day, his colleagues in Belgrade: „It will pass“, „big powers will make a deal“, „there are people who are on your side (he sent me a clip from a paper with an article written by a catholic priest)“, etc., promising me that at the close of bombardment, he will come to Belgrade at his own expenses and will give several lectures as help to his colleagues in Belgrade. Of course, he was true to his word. In 2000, he organized a Workshop (of course, with his associates from Europe: Grafarend, Sünkel, Hofmann-Wellenhof, and others) in Dubrovnik under the sponsorship of IAG, as an assistance to his colleagues of the South-East Europe. As soon as he reviewed the Serbian text of my monograph Least Squares he immediately made a proposal and financial aid for the book to be translated into English, etc.

I want to emphasize that Helmut Moritz‘s assistance to Vladeta Milovanović in 1967, the then assistant at the Belgrade University, in order to accept the theme of Ph. D. Thesis was important for the Serbian Geodesy. Milovanović's idea appeared contrary to the point of view of other European geodesists, but, he was supported by Helmut Moritz and Rudolf Sigl. After their support the topic of the thesis was accepted and finally it appeard that Milovanović was right. This launched the Serbian Geodesy into the European orbit.

*He gives assistance absolutely wholeheartedly*: An extract from Helmut‘s review of the Monograph Least Squares: „Must confess, my nerves are not in the best shape. So I worked under pressure and I was not so polite as I like to be. But I think I was able to help you with the English style, and help is more important than politeness". Even after my oral proposition he helps young colleagues from Serbia to complete their doctoral thesis in the European countries.

*He pays attention to everything and is observant of everything*: „that the examples be chosen carefully“, ... „that the book looks nicely“, ... „that the right theme has been selected in an article“, etc.

*In a specially sophisticated and correct way he pays attention*: „You are such an idealist …" - since I had to do everything concerning my book Least Squares myself, even to be its publisher; or, in reviewing the Least Squares: „English translation. Now since I know that the translator is responsible for the mistakes, I will correct much more severely and completely, without fear to offend you", …

Of course, there is *Helmut‘s humanitarian activity*: He is a member of the Humanists League in Sarajevo (Bosnia and Herzegovina); member of the Humanists League in Banja Luka (Republic of Srpska - Bosnia and Herzegovina), ...

*Helmut? Precise to the perfection*. A human having positive energy only. *For this real world he is completely unreal. He is exactly the person preached about in the Holy Scripture*!

I shall use saying of my – Serbian people: „*Helmut, God save you, and let you live another 100 years*“!, to the benefit of Science, and your friends, and, certainly, of your family.

## References

Personal communications with Helmut Moritz.