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Determination of the gravitational potential energy based on the Earth's 3D global density distribution

Introduction

Determination of the 3D density distribution δ inside the Earth's interior from the given external potential leads traditionally to the solution of improperly posed inverse problem of the gravitational potential. If the planet's gravitational potential energy E and the surface density are accepted as additional information, this problem transforms from an improperly posed to a properly posed problem with its possible solution for the 3D density $\delta(\rho, \vartheta, \lambda)$ through the three-dimensional Cartesian moments of a gravitating body (Mescheryakov, 1977). The gravitational potential energy E taken with the sign (–) represents the quadratic functional ($W = -E$) of δ (Moritz, 1990) and therefore can be applied to get a stable solution. This functional W represents the work of gravitation with remarkable expression for W through Dirichlet's integral $D(V,V)$ on the gravitational potential V being extended throughout all space (Thomson and Tait, 1879). As a result, additionally to standard definition of the energy E we need appropriate explicit relationships and accurate numerical E-values based on the 3D distribution of the Earth's density models.

One of possible approaches leads to the search of the stationary value E or the so-called Gauss' problem (Gauss, 1840). Gauss proved in his famous memoir (1840) that $W = -E$ has some minimal value $W_{\min}$. Thomson and Tait (1879) wrote, in particular: "The manner in which Gauss independently proved Green's theorems is more immediately and easily interpretable in terms of energy". The minimum amount of W is $W_{\min} = M \cdot V_0 / 2$, if all masses are concentrated on the boundary σ where the gravitational potential $V_0 = \text{const}$ and M is the Earth's mass (Idelson, 1932). A remarkable summary of the

Gauss' problem reads: „minimum and maximum potential energy correspond to physically (for the Earth) meaningless cases: a surface distribution and a mass point. The 'true' Earth lies somewhere in between" (Moritz, 1990). One of possible solutions for the answer about the energy E of the 'true' Earth was given by (Marchenko, 2007) in the form of inequality when the Gaussian distribution leads to a reliable estimation of the lower limit of $E=E_{\mathrm{Gauss}}$: various density models give E-values at the vicinity of E_{Gauss}.

In contrast to the last paper this study aims to derive according to Maxwell (1881) other kind of expression for the gravitational potential energy $E = -(W_{\min} + \Delta W)$. This representation allows a simple estimation of $W_{\min}$ and important treatment of the deviation ΔW from this minimal amount $W_{\min}$ as Dirichlet's integral on the internal potential V_i generated by an adopted density distribution. On the other hand, the Earth's mass and three principal moments of inertia are selected as initial information for the unique and exact solution of the restricted 3D Cartesian moments problem (Mescheryakov, 1991), preserving in this way the external gravitational potential from zero to second degree/order and the principal moments (A, B, C) of inertia. Thus, the 3D global density inside the spherical planet was adopted for the estimation of the potential energy E.

Basic relationships

Traditionally the computation of the Earth's gravitational potential energy is based on the following conventional expression

$$E = -W = -\frac{1}{2}\int_{\tau} V_i \cdot \delta \cdot d\tau \ , \tag{1}$$

where $\delta = \delta(r, \vartheta, \lambda)$ is the planet's volume density, V_i is the Earth's internal gravitational potential, τ is the planet's volume enclosed by the surface σ, and W is the work of gravitation required to transport the masses M from a state of infinite diffusion to their actual place inside the Earth.

However we prefer to use another treatment of Eq. (1) in terms of the internal potential only that leads to the equivalent and useful relationship for the energy E. Since our basic model of density can include density jumps and can represent some piecewise bounded function, let us suppose that $\delta \in L_2(\Sigma)$ is defined on the Hilbert space $L_2(\Sigma)$ of square-integrable functions inside the Earth's interior Σ. In this case the internal potential V_i has generalized second derivatives and satisfies Poisson's equation in almost all points of Σ (Holota, 1995):

$$\nabla^2 V_i = -4\pi G\delta \ , \ \left(\delta = -\frac{\nabla^2 V_i}{4\pi G}\right). \tag{2}$$

Substitution of Eq. (2) into Eq. (1) by means of the first Green's identity applied with (Maxwell, 1881) to Eq. (1) gives

$$W = -\frac{1}{8\pi G}\int_\tau V_i \cdot \nabla^2 V_i d\tau = -\frac{1}{8\pi G}\left[\int_\sigma V_i \frac{\partial V_i}{\partial n} d\sigma - \int_\tau D(V_i, V_i) d\tau\right], \tag{3}$$

$$\int_\tau D(V_i, V_i) d\tau = \int_\tau \left[\left(\frac{\partial V_i}{\partial x}\right)^2 + \left(\frac{\partial V_i}{\partial y}\right)^2 + \left(\frac{\partial V_i}{\partial z}\right)^2\right] d\tau = \int_\tau \left|\mathrm{grad} V_i\right|^2 d\tau \ . \tag{4}$$

The Green's transformation [Eq. (3)] is valid if we suppose that the function V_i and its first derivatives are continuous or even piecewise (Courant and Hilbert, 1951). Eq. (4) denotes always-nonnegative Dirichlet's integral. Note that Eq. (1) may be transformed via Green's identity as

$$4\pi \int_\tau V_i \cdot \delta \cdot d\tau = \int_\varpi D(V_i, V_i) d\tau \ , \tag{5}$$

where Dirichlet's integral is extended throughout all space ϖ . The interpretation of Dirichlet's integral in terms of the gravitational potential energy follows from Eq. (5).

Let us now assume that the boundary σ represents a level surface where the gravitational potential $V_i = V_0 = \mathrm{const}$. By this Eq. (3) gives

$$W = -\frac{1}{8\pi G}\left[V_0 \int_\sigma \frac{\partial V_i}{\partial n} d\sigma - \int_\tau D(V_i, V_i) d\tau\right] = \frac{V_0 M}{2} + \frac{1}{8\pi G}\int_\tau D(V_i, V_i) d\tau, \tag{6}$$

as a consequence of Gauss' theorem (Heiskanen and Moritz, 1967) applied to the first integral in the brackets of Eq. (6):

$$-4\pi GM = \int_\sigma \frac{\partial V_i}{\partial n} d\sigma \ . \tag{7}$$

According to Dirichlet's principle (*Kellogg*, 1929) the work $W=-E$ has some minimal value $W_{\min}$ if all masses are concentrated on the level surface σ when the gravitational potential $V_0=\text{const}$ and the interior is empty. In this case the internal potential $V_i=V_0=\text{const}$ represents the harmonic function inside the surface σ and leads to zero Dirichlet's integral in Eq. (6). Thus, the minimum amount of W becomes

$$W_{\min}=M\cdot V_0/2\ , \tag{8}$$

and represents the solution of the variational Gauss' problem (Idelson, 1932). Substitution of Eq. (8) into Eq. (6) leads to the following basic formulae

$$E=-(W_{\min}+\Delta W)\ ,\ \Delta W=\frac{1}{8\pi G}\int_\tau D(V_i,V_i)d\tau\ . \tag{9}$$

Thus, other kind of expression for *E*, given under the assumption that the boundary σ is a level surface, provides a simple estimation of $W_{\min}$ and remarkable treatment of the deviation ΔW from this minimal amount $W_{\min}$ as non-zero Dirichlet's integral when all masses are distributed inside τ according to an adopted density law.

Now we consider the application of Eq. (9) to the spherically symmetric density distribution $\delta=\delta(r)$ within the spherical Earth. In this case the gravitational potential $V_0=\text{const}$ will coincide with the potential of a point mass and the potential $V_0=GM/R$ of the surface distribution $\delta=\delta(R)=\text{const}$. By this Eq. (8) becomes

$$W_{\min}=GM^2/2R\ . \tag{10}$$

usually related to a homogeneous planet. The second term ΔW in Eq. (9) transforms to

$$\Delta W=\frac{4\pi}{8\pi G}\int_0^R\left(\frac{dV_i(r')}{dr'}\right)^2 r'^2dr'=\frac{1}{2G}\int_0^R\left(\frac{GM(r')}{r'^2}\right)^2 r'^2dr'=\frac{1}{2G}\int_0^R U^2dr', \tag{11}$$

where the gravity (gravitational attraction) $g(r)$ inside a stratified spherical Earth is expressed through the part of the Earth's mass $M(r)$:

$$\frac{GM(r)}{r^2}=-\frac{dV_i(r)}{dr}=g(r)\ , \tag{12}$$

$$M(r) = 4\pi \int_0^r \delta(r') \cdot r'^2 dr' \ . \tag{13}$$

Therefore, the gravitational potential energy of the spherically symmetric density distribution $\delta = \delta(r)$ becomes

$$E = -\left(\frac{GM^2}{2R} + \frac{1}{2G} \int_0^R \left(\frac{GM(r')}{r'} \right)^2 dr' \right) = -\left(\frac{GM^2}{2R} + \frac{1}{2G} \|U\|^2_{L_2[0,R]} \right). \tag{14}$$

The first term within brackets of Eq. (14) represents the minimal work of gravitation required to transport masses, having the total Earth's mass *M*, from a state of infinite diffusion onto the surface of a spherical planet with the radius *R*. Obviously, the mass $M(r)$ given by Eq. (13) represents the part of mass of the spherical Earth restricted by the radius *r*. In view of Eq. (9) or Eq. (11) the integral in the second term is bounded and can be treated through the norm $\|U\|_{L_2[0,R]}$ of the simple function $U(r) = GM(r)/r$ in the Hilbert space L_2 of square-integrable functions on the segment [0, *R*].

Estimation of the energy *E* based on the 3D density distribution

Let us consider according to Mescheryakov (1991) the mathematical model of the 3D global density distribution inside the spherical Earth having the radius *R* as the exact (restricted by the order 2) solution of the three-dimensional Cartesian moments problem rewritten here via the polar coordinates

$$\delta(\rho, \vartheta, \lambda) = K + \frac{\rho^2}{3R^2} \left(D_1 + D_2 P_{20}(\cos\vartheta) + \frac{D_3}{2} P_{22}(\cos\vartheta)\cos 2\lambda \right), \tag{15}$$

$$K = \frac{5}{4} \delta_m [5I_{000} - 7(I_{200} + I_{020} + I_{002})] = \frac{5}{8} \delta_m \left(10 - 7 \cdot \mathrm{Trace}(\mathbf{I})\right) \ , \tag{16}$$

$$D_1 = \frac{35}{8} \delta_m \left(5 \cdot \mathrm{Trace}(\mathbf{I}) - 6\right) \ , \tag{17}$$

$$D_2 = \frac{35}{2} \delta_m \left(A + B - 2C\right), \tag{18}$$

$$D_3 = \frac{35}{2}\delta_m\left(B - A\right) , \tag{19}$$

where δ_m is the mean density; $r = \rho \cdot R$ ($0 \le \rho \le 1$) is the distance from the origin of the coordinate system to a current point, ϑ and λ are the polar distance and longitude of this point.

Eq. (15) is given in the geocentric coordinate system of the principal axes of inertia ($\overline{A}$, $\overline{B}$, $\overline{C}$) and agreed with the Earth's mass M and all components of the Earth's tensor of inertia to preserve the external gravitational potential from zero to second degree/order and the principal moments of inertia. Mechanical parameters in Eq. (15) are expressed through the dimensionless Cartesian moments of the density of a gravitating body restricted by the order $n=p+q+r=2$:

$$I_{pqr}(\delta) = \frac{1}{MR^n}\int_\tau \delta x^p y^q z^r \, d\tau, \quad (p+q+r=n), \tag{20}$$

which for n=2 can be computed by means of the Earth's mass and the dimensionless principal moments of inertia A, B, and C normalized by the factor $1/MR^2$:

$$I_{000} = 1 \ , \ I_{200} = \frac{B+C-A}{2}, \ I_{020} = \frac{A-B+C}{2}, \ I_{002} = \frac{A+B-C}{2}, \tag{21}$$

where A, B, and C can be expressed via the 2nd-degree harmonic coefficients $\overline{A}_{20}$, $\overline{A}_{22}$ given in the principal axes system and the dynamical ellipticity H_D:

$$A = \sqrt{5}\overline{A}_{20} - \frac{\sqrt{15}\overline{A}_{22}}{3} - \frac{\sqrt{5}\overline{A}_{20}}{H_D} , \tag{22a}$$

$$B = \sqrt{5}\overline{A}_{20} + \frac{\sqrt{15}\overline{A}_{22}}{3} - \frac{\sqrt{5}\overline{A}_{20}}{H_D} , \tag{22b}$$

$$C = -\frac{\sqrt{5}\overline{A}_{20}}{H_D} . \tag{22c}$$

Eqs. (22) lead to the following relationship for the Trace(**I**) of the inertial tensor **I** or the mean moment $I_m = \mathrm{Trace}(\mathbf{I})/3$ of inertia

$$\mathrm{Trace}(\mathbf{I}) = (A + B + C) = 2(I_{200} + I_{020} + I_{002}) = \sqrt{5}\bar{A}_{20}\left(2 - \frac{3}{H_D}\right) = 3I_m. \quad (23)$$

Thus, in the above formulae *x*, *y*, *z* are the Cartesian coordinates of an internal point; $d\tau$ is the volume element of the sphere R ; $\bar{A}_{20}, \bar{A}_{22}$ are the fully normalized (non-zero) harmonic coefficients adopted here as Stokes constants in the Earth's principal axes system $O\bar{A}\bar{B}\bar{C}$. After transformation of Eq. (15) to the fully normalized associated Legendre's functions we get

$$\delta(\rho, \vartheta, \lambda) = \delta_{Roche}(\rho) + \Delta\delta_{3D}(\rho, \vartheta, \lambda), \quad (24)$$

where

$$\delta_{Roche}(\rho) = \frac{5}{8}\delta_m\left(10 - 7 \cdot \mathrm{Trace}(\mathbf{I})\right) + \rho^2 \frac{35}{24}\delta_m\left(5 \cdot \mathrm{Trace}(\mathbf{I}) - 6\right), \quad (25)$$

$$\Delta\delta_{3D}(\rho, \vartheta, \lambda) = \frac{35}{3}\rho^2\delta_m\left[\bar{A}_{20}\bar{P}_{20}(\cos\vartheta) + \bar{A}_{22}\bar{P}_{22}(\cos\vartheta)\cos 2\lambda\right]. \quad (26)$$

Eq. (25) represents the Roche's radial density (*Marchenko*, 2000):

$$\delta_{Roche}(\rho) = a + b\rho^2 , \quad (27)$$

$$a = \frac{5}{8}\delta_m\left(10 - 7 \cdot \mathrm{Trace}(\mathbf{I})\right), \quad (28)$$

$$b = \frac{35}{24}\delta_m\left(5 \cdot \mathrm{Trace}(\mathbf{I}) - 6\right), \quad (29)$$

where ($0 \le \rho \le 1$) is the relative distance from the origin of the coordinate system to a current point. Eq. (26) describes the deviation of the density distribution (24) from radial density profile, expressed only via 2nd degree harmonic coefficients of the external gravitational potential.

Initially we derive relationships for the internal potential V_i corresponding to this density law. Since the density (24) is separated on the two parts, the internal gravitational potential can be formed as

$$V_i = V_{Roche}(r) + V_{3D}(r,\vartheta,\lambda) = \int_\tau \frac{\delta_{Roche}(\rho)}{l} d\tau + \int_\tau \frac{\Delta\delta_{3D}(\rho,\vartheta,\lambda)}{l} d\tau \ , \quad (30)$$

where l is the reciprocal distance between a current points and the element of volume $d\tau$. According to (*Marchenko*, 2007) the internal potential $V_{Roche}(r)$ is expressed via coefficients given by Eqs. (28 – 29):

$$V_{Roche} = -\frac{\pi G}{15R^2}\left[10aR^2(r^2 - 3R^2) + 3b(r^4 - 5R^4)\right], \quad (31)$$

$$V_{Roche} = -\frac{\pi G\delta_m}{24R^2}\left[35 \cdot \mathrm{Trace}(\mathbf{I})\left(R^2 - r^2\right)^2 - 2\left(45R^4 - 50R^2r^2 + 21r^4\right)\right]. \quad (32)$$

The computation of $V_{3D}(r,\vartheta,\lambda)$ leads to the expansion of the inverse distance $1/l$ into convergent series for external and internal current points. Making necessary manipulations with the mentioned series of external and internal solid spherical harmonics and entering the orthogonality relations for spherical functions we finally get

$$V_{3D} = \frac{2\pi G\delta_m r^2(7R^2 - 5r^2)}{3R^2}\left(\overline{A}_{20}\overline{P}_{20}(\cos\vartheta) + \overline{A}_{22}\overline{P}_{22}(\cos\vartheta)\cos 2\lambda\right). \quad (33)$$

Then, applying Eq. (1) to the 3D density model [Eqs. (24 – 26)] and the corresponding internal potential given by Eqs. (30 – 33) we can find the expression for the estimation of the potential energy E of the spherical Earth. It has to be pointed out that the orthogonality relationships for spherical harmonics allow the direct computation of the gravitational potential energy E in the evident form

$$E = E_{Roche} + \Delta E_{3D} \ , \quad (34)$$

$$E_{Roche} = -\frac{1}{2}\int_\tau \delta_{Roche}(r)V_{Roche}(r)d\tau \ , \quad (35)$$

$$\Delta E_{3D} = -\frac{1}{2}\int_{\tau} \Delta\delta_{3D}(r,\vartheta,\lambda)V_{3D}(r,\vartheta,\lambda)d\tau \ . \tag{36}$$

By this, after integration Eqs. (35 – 36) become

$$E_{Roche} = -\frac{5}{81}\pi^2 G\delta_m^2 R^5\left(7\cdot\mathrm{Trace}(\mathbf{I})^2 - 24\mathrm{Trace}(\mathbf{I}) + 36\right), \tag{37}$$

$$\Delta E_{3D} = -\frac{560}{81}\pi^2 G\delta_m^2 R^5 k_2 \ , \ k_2 = \overline{A}_{20}^2 + \overline{A}_{22}^2 = \sum_{m=0}^{2}(\overline{C}_{2m}^2 + \overline{S}_{2m}^2) \ , \tag{38}$$

where k_2 is the degree variance of the second order given initially in the principal axes system. It should be noted that k_2 represents the invariant characteristics of the Earth's gravity field, which are independent of linear transformations of the coordinate system and can be computed in the standard Earth's-fixed coordinate system. Therefore, the gravitational potential energy of the considered 3D density model as restricted solution of the Cartesian moments problem allows the expression for E consisting of two basic constituents. The first component, E_{Roche}, corresponds to the energy of the radial density distribution. The second component, ΔE_{3D}, is linked to the external potential harmonic coefficients of the 2nd degree/order describing in this way the contribution of global lateral density heterogeneities.

Model	$W_{\min}$	ΔW	ΔW_{3D}	E
Rubincam, 1979	–	–	–	–2.45
Homogeneous Earth	1.8687	0.3737	–	–2.2425
Legendre-Laplace model	1.8687	0.5965	–	–2.4652
Roche's model	1.8687	0.6101	–	–2.4788
Bullard's model	1.8687	0.6132	–	–2.4820
Gauss' model	1.8687	0.6386	–	–2.5073
PREM model	1.8685	0.6199	–	–2.4884
This study	1.8687	0.6101	3.41×10^{-6}	–2.4788

Table 1: Estimations of the gravitational potential energy E [ergs×10^{39}]

Therefore, according to the above mentioned approach and the basic equation (9) we finally get

$$E = -\left(W_{\min} + \Delta W_{Roche} + \Delta W_{3D}\right), \tag{39}$$

$$W_{\min} = 8\pi^2 G R^5 \delta_m^2 / 9 = GM^2 / 2R , \tag{40}$$

$$\Delta W_{Roche} = \frac{GM^2}{144R}\left[35\mathrm{Trace}(\mathbf{I})^2 - 120\mathrm{Trace}(\mathbf{I}) + 108\right], \tag{41}$$

$$\Delta W_{3D} = -\Delta E_{3D} = \frac{35GM^2}{9R} k_2 . \tag{42}$$

It is easy to verify that $W_{Roche} + \Delta W_{3D}$ correspond to the second term in Eq. (9) because ΔW_{3D} gives zero contribution in the first term $W_{\min}$. With $\overline{A}_{20} = -484.1692924 \cdot 10^{-6}$, $\overline{A}_{22} = 2.8127085 \cdot 10^{-6}$, $\delta_m = 5.51483\,\mathrm{g/cm^3}$, $G = 6.67259 \cdot 10^{-11}\ \mathrm{m^3 kg^{-1} s^{-2}}$, and R=6371 km the computation of the each term in Eq. (39) is straightforward. Table 1 lists estimations of the gravitational potential energy E for the well known radial density laws and E-estimate given by (Rubincam, 1979) for the spherical Earth differentiated into present-day core/mantle and considered as two homogeneous shells with different values of the mean density.

Thus, we get a suitable accordance between E-estimates derived from different continuous density distributions, the most widely used piecewise PREM model, and E-value from the discussed 3D continuous density as restricted solution of the three-dimensional Cartesian moments problem inside a spherical planet. All E-estimates are close to E=E_{Gauss} of Gaussian distribution that can be considered as reliable estimation of the lower limit of the gravitational potential energy E. It is evident that the contribution of lateral density heterogeneities ΔE_{3D} (corresponded to the adopted 3D density model) into E-value is insignificant in relation to the radial component (Table 1). According to Eq. (39) the estimation of the gravitational potential energy of a spherical planet can be made within three steps. The first one corresponds to the calculation of the minimal amount of the work $W_{\min}$. On the second step we determine the work $W_{\min} + \Delta W_{Roche}$ of gravitation to transport the masses M inside the spherically symmetric Earth. The third step involves the influence of global lateral density heterogeneities and gives the work of gravitation $W_{\min} + \Delta W_{Roche} + \Delta W_{3D}$ to transport the masses M from a state of infinite diffusion to their 3D condition inside the planet.

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