Friedrich Robert Helmert:
seine Lösung des Anholonomitäts-Problem der Geodäsie
oder
warum ist Geodäsie physikalisch?
Von Gauss – Listing über Bruns zu Molodenski:
geometrische und physikalische Höhen und Höhensysteme:
global versus lokal

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Friedrich Robert Helmert

geboren in Freiberg in Sachsen
Am 31. Juli 1843
(Gedenktafel 1913, erneuert 1967)

geodätische Vorläufer: C.F. Gauss,
Soldner (Bayern), Bohnenberger (Württemberg), Baeyer (Preußen),
Gerling (Hessen), Hansen (Sachsen-Coburg)

(i) Bürgerschule in Freiberg

(ii) Ab 14 Jahren: St. Annen-Realschule in Dresden

(iii) Mit 16 Jahren: Polytechnische Schule/TH Dresden

Lehrer: August Nagel/Lehrstuhl Geodäsie

(iv)Assistent von A. Nagel,

Reisestipendium auf grund von herausragenden Leistungen

Gradmessung – Triangulation von Sachsen

(v) Ergänzungsstudium Mathe/Physik
in Leipzig (2 Jahre)

(vi) 1868: Dissertation

„Studium über rationelle Vermessungen im Gebiet der Höheren Geodäsie“

(vii) 1869: Observator Sternwarte Hamburg

1874: erste Publikation

„Vermessung und rechnerische Ausgleichung eines Sternhaufens“

(viii) 1870: Berufung auf die ordentliche Professur „Geodäsie“ an der Polytechnischen Schule/Universität Aachen mit 29 Jahren!

2 fundamentale Werke:

A) Ausgleichungsrechnung nach der Methode der kleinsten Quadrate mit Anwendungen auf die Geodäsie und die Theorie: 1872

B) Die mathematischen und physikalischen Theorien der Höheren Geodäsie: 1880, 1884 (zwei Bände)

Helmert's Arbeitsthemen:

- *Math. Statistik und Ausgleichungsrechnung*

  + HELMERT TRANSFORMATION

    überbestimmtes Ausgleichungsproblem:
    Translation Rotation, Maßstab, „Lineare Gruppe“

  + CHI-QUADRAT VERTEILUNG

  + GAUSS-HELMERT MODEL

    Bedingungsgleichungen mit Unbekannten

  + Var-Kovarianz Schätzung

    „Varianz Komponenten-Schätzer, welche positiv sind“

  + CHOLESKI FAKTORISIERUNG

  + STRENGE AUSGLEICHUNG des EUROPÄISCHEN DREIECKSNETZES

- *Physikalische Geodäsie*

  + HELMERT Projektion
  + HELMERT‘s ELLIPSOID - Übergang
  + HELMERT NIVEAUELLIPSOID
  + HELMERT HÖHEN
  + HELMERT LOTABWEICHUNGEN

    „Defintion“
  + POLHÖHEN Änderungen
  + GEODYNAMIK

„HELMERT“: HOLONOM versus ANHOLONOM:

? Warum ist Geodäsie physikalisch ?

F.R. HELMERT hatte auf Rat seines Doktorvaters zwei Jahre Physik in Leipzig studiert:

! FROBENIUS LEMMA !

„integrierende Funktionen“

$$\oint dW = -\oint \Gamma \not{d}H = 0 \ \text{ versus } \ \oint \not{d}H \neq 0 \ : \ \text{M. PLANCK}$$

„RING-INTEGRAL“: geschlossener Weg

„W ist das GAUSS-GREEN Potential“

CARTAN Differential-Geometry: „CARTAN calculus“

In einem mitrotierenden Koordinaten-System gibt es 4 Typen von Kräften:

(i) Gravitation, (ii) Zentrifugalkraft: KONSERVATIV

(iii) EULER („Drehimpuls“: Polbewegung, Tageslängenänderung oder Präzession/Nutation) NICHT-KONSERVATIV

(iv) CORIOLIS („Strömungen“): NICHT-KONSERVATIV

$$\text{(ii) } \operatorname{div}\operatorname{grad} W = -4\pi G\rho + 2\Omega^2$$

$$\text{KONSERVATIV: (i) } \operatorname{grad} W = \operatorname{grad} U + \operatorname{grad} V$$

$$\text{NICHT-KONSERVATIV: (iii) } \operatorname{rot} \Gamma = 2\Omega^{\bullet}$$ (starrer Körper, der rotiert)

BASIS: DIFFERENTIAL FORMEN

„EXTERIOR CALCULUS“: ELIE CARTAN

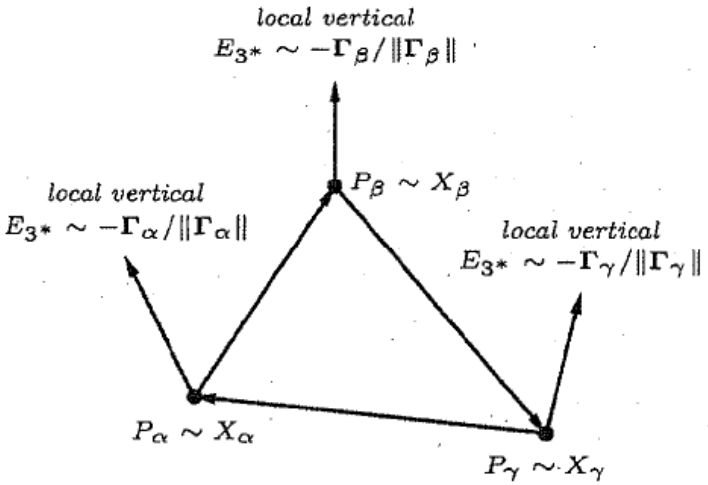
„HELMERT“:

2 Beispiele

ZfV 122 (1987) 413 – 424:

„Der Einfluß der Lotrichtung auf lokale geodätische Netze“

BEISPIEL:



Triangular network $\{P_\alpha, P_\beta, P_\gamma | \mathbf{O}\}$, placement vectors at the origin $\mathbf{O}$, local verticals $E_3(P_\alpha)$, $E_3(P_\beta)$, $E_3(P_\gamma)$, Γ_α, Γ_β, Γ_γ local gravity vectors

$$\overset{\alpha}{X}_{\alpha\beta} := \overset{\alpha}{X}_\beta - \overset{\alpha}{X}_\alpha, \ldots, \overset{\alpha}{Z}_{\gamma\alpha} := \overset{\alpha}{Z}_\alpha - \overset{\alpha}{Z}_\gamma$$

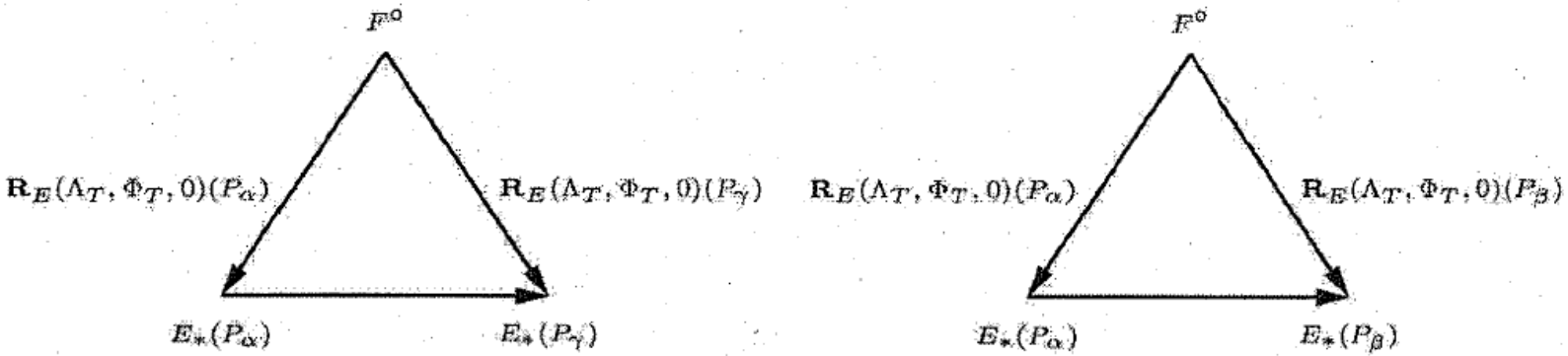
$$\overset{\alpha}{X}_{\alpha\beta} + \overset{\alpha}{X}_{\beta\gamma} + \overset{\alpha}{X}_{\gamma\alpha} = 0$$

$$\overset{\alpha}{Y}_{\alpha\beta} + \overset{\alpha}{Y}_{\beta\gamma} + \overset{\alpha}{Y}_{\gamma\alpha} = 0$$

$$\overset{\alpha}{Z}_{\alpha\beta} + \overset{\alpha}{Z}_{\beta\gamma} + \overset{\alpha}{Z}_{\gamma\alpha} = 0$$

Holonomity condition in terms of relative coordinates in a fixed reference system, fixed to the reference point P_α

BEISPIEL:



Commutative diagrams: *moving horizon reference systems* E_* versus *fixed equatorial reference systems* F°

E. GRAFAREND: ZfV 122 (1987) 413-424

"anholonomity condition"

$$\overset{\alpha}{X}_{\alpha\beta} := \overset{\alpha}{X}_{\beta} - \overset{\alpha}{X}_{\alpha}, \ldots, \overset{\alpha}{Z}_{\gamma\alpha} := \overset{\alpha}{Z}_{\alpha} - \overset{\alpha}{Z}_{\gamma}$$

$$\overset{\alpha}{X}_{\alpha\beta} + \overset{\beta}{X}_{\beta\gamma} + \overset{\gamma}{X}_{\gamma\alpha} \neq 0 \quad \text{(misclosure)}$$

$$\overset{\alpha}{Y}_{\alpha\beta} + \overset{\beta}{Y}_{\beta\gamma} + \overset{\gamma}{Y}_{\gamma\alpha} \neq 0 \quad \text{(misclosure)}$$

$$\overset{\alpha}{Z}_{\alpha\beta} + \overset{\beta}{Z}_{\beta\gamma} + \overset{\gamma}{Z}_{\gamma\alpha} \neq 0 \quad \text{(misclosure)}$$

"representation of the base vectors in the horizon reference frame E^* "

$$\mathbf{E}_{1^*} X_{\alpha\beta} + \mathbf{E}_{2^*} Y_{\alpha\beta} + \mathbf{E}_{3^*} Z_{\alpha\beta}$$

"Direct and inverse transformation of Cartesian coordinates into spherical coordinates"

(horizontal coordinate $H_{\alpha\beta}$, vertical coordinate $V_{\alpha\beta}$, distance S, azimuth $A_{\alpha\beta}$, vertical angle $B_{\alpha\beta}$, horizontal orientiation unknown)

$$X_{\alpha\beta} = S_{\alpha\beta} \cos A_{\alpha\beta} \cos B_{\alpha\beta},$$
$$Y_{\alpha\beta} = S_{\alpha\beta} \sin A_{\alpha\beta} \cos B_{\alpha\beta},$$
$$Z_{\alpha\beta} = S_{\alpha\beta} \sin B_{\alpha\beta}$$
$$A_{\alpha\beta} := H_{\alpha\beta} + \Sigma_{\alpha} = \arctan\left(Y_{\alpha\beta}/X_{\alpha\beta}\right),$$
$$B_{\alpha\beta} := V_{\alpha\beta} = \arctan\left(Z_{\alpha\beta}/\sqrt{X_{\alpha\beta} + Y_{\alpha\beta}}\right)$$
$$S_{\alpha\beta} := \sqrt{X^2_{\alpha\beta} + Y^2_{\alpha\beta} + Z^2_{\alpha\beta}}$$

Point transformation
$\mathbf{E}^{\alpha}_{*} \rightarrow \mathbf{E}^{\beta}_{*}$ *and* $\mathbf{E}^{\beta}_{*} \rightarrow \mathbf{E}^{\gamma}_{*}$

"Euler angles"

$$\mathbf{E}^{\alpha}(P_{\alpha}) \rightarrow \mathbf{E}^{\beta}(P_{\beta}):$$
$$\mathbf{E}^{\alpha}(P_{\alpha})\mathbf{R}_E(\Lambda_{\alpha}, \Phi_{\alpha}, 0)\mathbf{R}^T_E(\Lambda_{\beta}, \Phi_{\beta}, 0)$$
$$\Delta\Lambda := \Lambda_{\beta} - \Lambda_{\alpha}, \quad \Delta\Phi := \Phi_{\beta} - \Phi_{\alpha}$$
$$\mathbf{R}_E(\Lambda_{\alpha}, \Phi_{\alpha}, 0)\mathbf{R}^T_E(\Lambda_{\beta}, \Phi_{\beta}, 0)$$
$$\doteq \begin{bmatrix} 1 & -\Delta\Lambda \sin\Phi_{\alpha} & \Delta\Phi \\ \Delta\Lambda \sin\Phi_{\alpha} & 1 & \Delta\Lambda \cos\Phi_{\alpha} \\ \Delta\Phi & -\Delta\Lambda \cos\Phi_{\alpha} & 1 \end{bmatrix}$$

"antisymmetric matrix $\mathbf{A}$*"*

$$-\begin{bmatrix} \overset{\beta}{X}_{\beta\alpha} \\ \overset{\beta}{Y}_{\beta\alpha} \\ \overset{\beta}{Z}_{\beta\alpha} \end{bmatrix} = (\mathbf{I} - \mathbf{A}^T) \begin{bmatrix} \overset{\alpha}{X}_{\alpha\beta} \\ \overset{\alpha}{Y}_{\alpha\beta} \\ \overset{\alpha}{Z}_{\alpha\beta} \end{bmatrix}$$

analogue formulae for

$$X_{\gamma\alpha}, \ldots, Z_{\gamma\alpha}$$

analogue formulae for

$$A_{\beta\alpha},\ B_{\beta\alpha},\ S_{\beta\alpha},\ \ldots,\ A_{\gamma\alpha},\ B_{\gamma\alpha},\ S_{\gamma\alpha}$$

Anholonomity in a moving frame at points $\{P_{\alpha}, P_{\beta}, P_{\gamma}\}$

Erstes Beispiel

$$\overset{\alpha}{X}_{\alpha\beta} = +30\ m, \quad \overset{\alpha}{X}_{\beta\gamma} = +50\ m, \quad \overset{\alpha}{X}_{\gamma\alpha} = -80\ m$$

$$\overset{\alpha}{Y}_{\alpha\beta} = +30\ m, \quad \overset{\alpha}{Y}_{\beta\gamma} = -50\ m, \quad \overset{\alpha}{Y}_{\gamma\alpha} = +20\ m$$

$$\overset{\alpha}{Z}_{\alpha\beta} = +5\ m, \quad \overset{\alpha}{Z}_{\beta\gamma} = +15\ m, \quad \overset{\alpha}{Z}_{\gamma\alpha} = -20\ m$$

$$\Lambda_{\alpha\beta} = 1'' \sim 4.85 \cdot 10^{-6}\ \text{RAD}, \quad \Phi_{\alpha\beta} = -0.5'' \sim -2.42 \cdot 10^{-6}\ \text{RAD}$$

$$\Lambda_{\alpha\gamma} = -1'' \sim -4.85 \cdot 10^{-6}\ \text{RAD}, \quad \Phi_{\alpha\gamma} = -2.5'' \sim -12.12 \cdot 10^{-6}\ \text{RAD}$$

$$\Phi_{\alpha} = 48.783^{\circ}$$

25 meter local network

$$\begin{bmatrix} {}^{\beta}X_{\beta\gamma} \\ {}^{\beta}Y_{\beta\gamma} \\ {}^{\beta}Z_{\beta\gamma} \end{bmatrix} = \begin{bmatrix} 1 & +\Lambda_{\alpha\beta}\sin\Phi_{\alpha} & +\Phi_{\alpha\beta} \\ -\Lambda_{\alpha\beta} & 1 & -\Lambda_{\alpha\beta}\cos\Phi_{\alpha} \\ -\Phi_{\alpha\beta} & +\Lambda_{\alpha\beta}\cos\Phi\alpha & 1 \end{bmatrix} \begin{bmatrix} {}^{\alpha}X_{\beta\gamma} \\ {}^{\alpha}Y_{\beta\gamma} \\ {}^{\alpha}Z_{\beta\gamma} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & +3.65\cdot 10^{-6} & -2.42\cdot 10^{-6} \\ -3.65\cdot 10^{-6} & 1 & -3.19\cdot 10^{-6} \\ +2.42\cdot 10^{-6} & +3.19\cdot 10^{-6} & 1 \end{bmatrix} \begin{bmatrix} +50\ m \\ -50\ m \\ +15\ m \end{bmatrix}$$

$$\begin{bmatrix} {}^{\gamma}X_{\gamma\alpha} \\ {}^{\gamma}Y_{\gamma\alpha} \\ {}^{\gamma}Z_{\gamma\alpha} \end{bmatrix} = \begin{bmatrix} 1 & +\Lambda_{\alpha\gamma}\sin\Phi_{\alpha} & +\Phi_{\alpha\gamma} \\ -\Lambda_{\alpha\gamma} & 1 & -\Lambda_{\alpha\gamma}\cos\Phi_{\alpha} \\ -\Phi_{\alpha\gamma} & +\Lambda_{\alpha\gamma}\cos\Phi\alpha & 1 \end{bmatrix} \begin{bmatrix} {}^{\alpha}X_{\gamma\alpha} \\ {}^{\alpha}Y_{\gamma\alpha} \\ {}^{\alpha}Z_{\gamma\alpha} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & +-3.65\cdot 10^{-6} & -12.12\cdot 10^{-6} \\ +3.65\cdot 10^{-6} & 1 & +3.19\cdot 10^{-6} \\ +12.12\cdot 10^{-6} & -3.19\cdot 10^{-6} & 1 \end{bmatrix} \begin{bmatrix} -80\ m \\ +20\ m \\ -20\ m \end{bmatrix}$$

25 meter local network, detailed computation

$$\overset{\beta}{X}_{\beta\gamma} = +50\ m - 0.22\ mm, \quad \overset{\gamma}{X}_{\gamma\alpha} = -80\ m + 0.17\ mm$$

$$\overset{\beta}{Y}_{\beta\gamma} = -50\ m - 0.23\ mm, \quad \overset{\gamma}{Y}_{\gamma\alpha} = +20\ m - 0.36\ mm$$

$$\overset{\beta}{Z}_{\beta\gamma} = +15\ m - 0.04\ mm, \quad \overset{\gamma}{Z}_{\gamma\alpha} = -20\ m - 1.03\ mm$$

$$\overset{\alpha}{X}_{\alpha\beta} + \overset{\beta}{X}_{\beta\gamma} + \overset{\gamma}{X}_{\gamma\alpha} \neq 0 : \quad -0.22\ mm + 0.17\ mm = -0.05\ mm$$

$$\overset{\alpha}{Y}_{\alpha\beta} + \overset{\beta}{Y}_{\beta\gamma} + \overset{\gamma}{Y}_{\gamma\alpha} \neq 0 : \quad -0.23\ mm - 0.35\ mm = -0.59\ mm$$

$$\overset{\alpha}{Z}_{\alpha\beta} + \overset{\beta}{Z}_{\beta\gamma} + \overset{\gamma}{Z}_{\gamma\alpha} \neq 0 : \quad -0.04\ mm - 1.03\ mm = -1.07\ mm$$

25 meter local network, misclosures

E. GRAFAREND: ZfV 122 (1987) 413-424

Zweites Beispiel

$$\overset{\alpha}{X}_{\alpha\beta} = +500\ m, \quad \overset{\alpha}{X}_{\beta\gamma} = +800\ m, \quad \overset{\alpha}{X}_{\gamma\alpha} = -1300\ m$$

$$\overset{\alpha}{Y}_{\alpha\beta} = +500\ m, \quad \overset{\alpha}{Y}_{\beta\gamma} = -800\ m, \quad \overset{\alpha}{Y}_{\gamma\alpha} = +300\ m$$

$$\overset{\alpha}{Z}_{\alpha\beta} = +50\ m, \quad \overset{\alpha}{Z}_{\beta\gamma} = +150\ m, \quad \overset{\alpha}{Z}_{\gamma\alpha} = -200\ m$$

$$\Lambda_{\alpha\beta} = 25'' \sim 12.12 \cdot 10^{-5}\ \text{RAD}, \quad \Phi_{\alpha\beta} = -15'' \sim -7.27 \cdot 10^{-5}\ \text{RAD}$$

$$\Lambda_{\alpha\gamma} = -15'' \sim -7.27 \cdot 10^{-5}\ \text{RAD}, \quad \Phi_{\alpha\gamma} = -45'' \sim -2.18 \cdot 10^{-4}\ \text{RAD}$$

$$\Phi_{\alpha} = 48.783^{\circ}$$

500 meter local network

$$\begin{bmatrix} {}^{\beta}X_{\beta\gamma} \\ {}^{\beta}Y_{\beta\gamma} \\ {}^{\beta}Z_{\beta\gamma} \end{bmatrix} = \begin{bmatrix} 1 & +\Lambda_{\alpha\beta}\sin\Phi_{\alpha} & +\Phi_{\alpha\beta} \\ -\Lambda_{\alpha\beta} & 1 & -\Lambda_{\alpha\beta}\cos\Phi_{\alpha} \\ -\Phi_{\alpha\beta} & +\Lambda_{\alpha\beta}\cos\Phi\alpha & 1 \end{bmatrix} \begin{bmatrix} {}^{\alpha}X_{\beta\gamma} \\ {}^{\alpha}Y_{\beta\gamma} \\ {}^{\alpha}Z_{\beta\gamma} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & +9.12\cdot 10^{-5} & -7.27\cdot 10^{-5} \\ -9.12\cdot 10^{-5} & 1 & -7.99\cdot 10^{-5} \\ +7.27\cdot 10^{-5} & +7.99\cdot 10^{-5} & 1 \end{bmatrix} \begin{bmatrix} +800\ m \\ -800\ m \\ +150\ m \end{bmatrix}$$

$$\begin{bmatrix} {}^{\gamma}X_{\gamma\alpha} \\ {}^{\gamma}Y_{\gamma\alpha} \\ {}^{\gamma}Z_{\gamma\alpha} \end{bmatrix} = \begin{bmatrix} 1 & +\Lambda_{\alpha\gamma}\sin\Phi_{\alpha} & +\Phi_{\alpha\gamma} \\ -\Lambda_{\alpha\gamma} & 1 & -\Lambda_{\alpha\gamma}\cos\Phi_{\alpha} \\ -\Phi_{\alpha\gamma} & +\Lambda_{\alpha\gamma}\cos\Phi\alpha & 1 \end{bmatrix} \begin{bmatrix} {}^{\alpha}X_{\gamma\alpha} \\ {}^{\alpha}Y_{\gamma\alpha} \\ {}^{\alpha}Z_{\gamma\alpha} \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & +-5.47\cdot 10^{-5} & -2.18\cdot 10^{-4} \\ +5.47\cdot 10^{-5} & 1 & +4.79\cdot 10^{-5} \\ +2.18\cdot 10^{-4} & -4.79\cdot 10^{-5} & 1 \end{bmatrix} \begin{bmatrix} -1300\ m \\ +300\ m \\ -200\ m \end{bmatrix}$$

500 meter local network, detailed computation

$$\overset{\beta}{X}_{\beta\gamma} = +800\ m - 83.9\ mm, \quad \overset{\gamma}{X}_{\gamma\alpha} = -1300\ m + 27.2\ mm$$

$$\overset{\beta}{Y}_{\beta\gamma} = -800\ m - 84.9\ mm, \quad \overset{\gamma}{Y}_{\gamma\alpha} = +300\ m - 80.7\ mm$$

$$\overset{\beta}{Z}_{\beta\gamma} = +150\ m - 5.8\ mm, \quad \overset{\gamma}{Z}_{\gamma\alpha} = -200\ m - 297.8\ mm$$

$$\overset{\alpha}{X}_{\alpha\beta} + \overset{\beta}{X}_{\beta\gamma} + \overset{\gamma}{X}_{\gamma\alpha} \neq 0: \quad -83.9\ mm + 27.2\ mm = -56.7\ mm$$

$$\overset{\alpha}{Y}_{\alpha\beta} + \overset{\beta}{Y}_{\beta\gamma} + \overset{\gamma}{Y}_{\gamma\alpha} \neq 0: \quad -84.9\ mm - 80.7\ mm = -165.6\ mm$$

$$\overset{\alpha}{Z}_{\alpha\beta} + \overset{\beta}{Z}_{\beta\gamma} + \overset{\gamma}{Z}_{\gamma\alpha} \neq 0: \quad -5.8\ mm - 297.8\ mm = -303.6\ mm$$

500 meter local network, misclosures

BEISPIEL für ANHOLONOMITÄT

H. MORITZ: 3 OSU – Reports 1978

ROTATION – DREHIMPULS

H. MORITZ und I.I. MUELLER (1987): Earth rotation, UNGAR, New York

(i) Kinematische EULER-Gleichungen

$\rightarrow \omega = \mathrm{F}^{-1}(\mathrm{d}\lambda, \mathrm{d}\theta, \mathrm{d}\psi)$

„EULER oder CARDAN Winkel“

$\leftarrow (\mathrm{d}\lambda, \mathrm{d}\theta, \mathrm{d}\psi) = \mathrm{F}\omega$

„FROBENIUS Matrix
von integrierenden Faktoren“

„ω Rotationskomponenten: CARTAN“

(ii) Dynamische EULER-Gleichungen

„gewichtete EULER-Gleichungen“

„Trägheitsmoment: symmetrische 3x3-Matrix“

HEINRICH BRUNS

* 4. Sept. 1848 Berlin, gest. 23. Sept. 1919

MATHEMATIKER – ASTRONOM – GEODÄSIE

* Studium Mathematik, Astronomie und Physik in Berlin/Universität

LEHRER: KUMMER – WEIERSTRASS

1872 – 1973: Sternwarte PULMOWA

Bis 1878: Sternwarte DORPAT / Dozent an der Universität

1878: ausserord. Prof. für Mathematik: BERLIN

1882: **ord. Prof. für Astronomie/Sternwarte Leipzig**
REKTOR

in Berlin war er am Geodätische Institut tätig.

Hauptaufgaben

- elliptische Integrale (DORPAT 1875)
- Figur der Erde (Berlin 1878)
- Astronomische Refraktion
- Integrale des Vielkörperproblems: 3 Körperproblem, ? Stabilität ? (1887: LEIPZIG)
- EIKONAL (1895)

  geometrische Optik / 1. FRESNEL-ZONE

Potential-Theorie, GLEICHGEWICHTSFIGUREN

* BRUNSscher Polyeder

* Randwertaufgabe relativ zur Referenzkugel

BRUNS → STOKES → VENING-MEINESZ

VORTRÄGE von H. KAUTZLEBEN und E. BUSCHMANN

„HÖHENSYSTEME“

(i) Dividiere das Potential/Potentialunterschiede durch eine Konstante, z.B. dem Mittelwert des Vermessungsgebietes

$$\frac{\mathrm{d}W}{\Gamma(\mathrm{mean})}$$

(ii) Dividiere das Potential/Potentialunterschiede durch die SOMIGLIANA-PIZZETTI Schwere in Reihen entwickelt (NANO - GAL)

„Ellipsoidisches Koordinaten-System“

$$\frac{\mathrm{d}W}{\Gamma(\mathrm{SOM}-\mathrm{PI})}$$

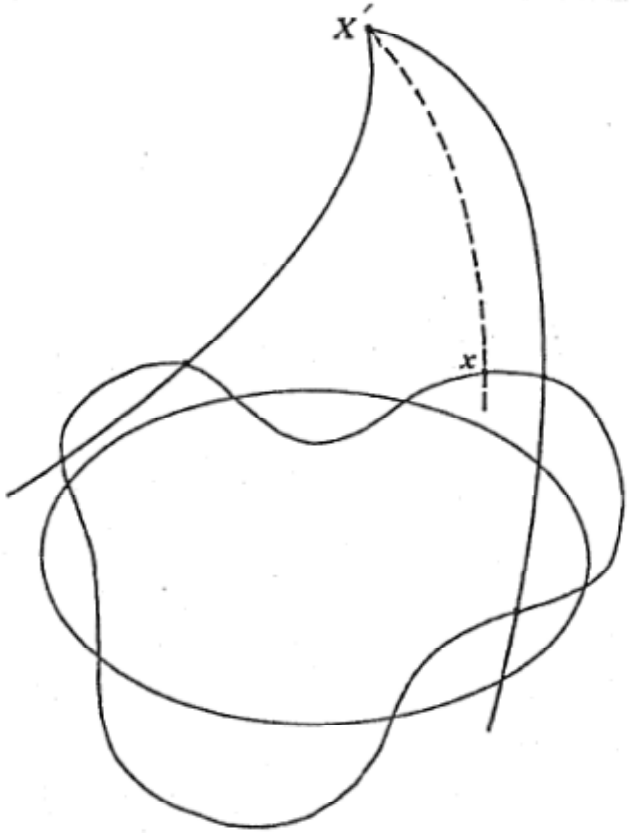
A. ARDALAN und E: GRAFAREND

(NANO-GAL: J. GEODESY **75** (2001) 424-437)

Field lines of gravity, their curvature and torsion, the Lagrange and the Hamilton equations of the plumbline

The length of the gravitational field lines/of the orthogonal trajectories of a family of gravity equipotential surfaces/of the plumbline between a terrestrial topographic point and a point on a reference equipotential surface like the geoid – also known as the orthometric height – plays a central role in Satellite Geodesy as well as in Physical Geodesy. As soon as we determine the geometry of the Earth pointwise by means of a satellite GPS (Global Positioning System: «global problem solver») we are left with the problem of converting ellipsoidal heights (geometric heights) into orthometric heights (physical heights). For the computation of the plumbline we derive its three differential equations of first order as well as the three geodesic equations of second order. The three differential equations of second order take the form of a Newton differential equation when we introduce the parameter time via the Marussi gauge on a conformally flat three-dimensional Riemann manifold and the generalized force field, the gradient of the superpotential, namely the modulus of gravity squared and taken half. In particular, we compute curvature and torsion of the plumbline and prove their functional relationship to the second and third derivatives of the gravity potential. For a spherically symmetric gravity field, curvature and torsion of the plumbline are zero, the plumbline is straight. Finally we derive the three Lagrangean as well as the six Hamiltonian differential equations of the plumbline, in particular in their star form with respect to Marussi gauge.

Erik W. Grafarend (1997): Annali di Geofisica **40** (1997) 1233 - 1247



Projective heights in gravity space, minimal distance mapping with respect to a reference equipotential surface, the *geoid* $w_0 = (62,636,856.5 \pm 3)\,\mathrm{ms}^{-2}$, *at a reference epoch* t_0, *orthometric height* h_G (length of the plumbline / orthogonal trajectory from $X \in T^2$ to $x \in$ GEOID), representation of the *geoid* with respect to an ellipsoid of revolution $E^2_{a,b}$ according to J. ENGELS and E. GRAFAREND (1992a, b)

Duality between horizontal and vertical fields in $\{\mathbb{R}^3, \delta_{ij}\}$ equipped with a Euclidean metric δ_{ij}.

Normalized tangent vector of the plumbline is identical to the surface normal vector of an equipotential surface

1st order differential equations

$$\frac{dx}{dS} = -\operatorname{grad} w / \| \operatorname{grad} w \| \sim \frac{dx^k}{dS} = -\partial_k w / \sqrt{\delta^{lm} \partial_l w \partial_m w}$$

2nd order differential equations

$$\frac{d^2 x^k}{dS^2} + \frac{\gamma, l}{\gamma^3} (\gamma^2 \delta^{kl} - \gamma^k \gamma^l) = 0$$

Marussi gauge

$$\| x^\bullet \| = \| \operatorname{grad} w \|, \quad \frac{dx}{dS} = \| x^\bullet \|^{-1} x^\bullet$$

1st order differential equation of the plumbline in Marussi gauge

$$x^\bullet = -\operatorname{grad} w$$

2nd order differential equation of the plumbline in Marussi gauge

$$x^{\bullet\bullet k} = -(\partial_l \gamma^k) x^{\bullet l} = (\partial_l \gamma^k) \partial_l w = (\partial_l \gamma^k) \gamma^l$$

$$x^{\bullet\bullet k} - \frac{1}{2} \partial_k \gamma^2 (x^m) = 0$$

Molodensky potential telluroid based on a minimum-distance map. Case study: the quasi-geoid of East Germany in the World Geodetic Datum 2000

The telluroid as introduced by M.S. Molodensky (1945, 1948, 1960) may be considered the best analytical representation of the irregular surface of the Earth. Given the placement vector of a point in geometry space, for instance by GPS (global problem solver), and reference gravity potential in gravity space, the telluroid can be uniquely defined by a properly chosen projection. For instance, astronomical longitude/astronomical latitude (spherical coordinates in gravity space) at a topographic point could be defined to coincide with reference longitude/reference latitude (spherical coordinates in reference gravity space) at a telluroid point in order to establish an isoparametric mapping of the telluroid. Bode and Grafarend (1982) extensively studied such an isoparametric telluroid mapping with respect to a reference gravity potential which is additively decomposed into (1) the zero-order coefficient of a spherical harmonic expansion of the gravitational potential and (2) the centrifugal potential. In particular, they succeeded in identifying the singular points of such a telluroid map. Here we aim at an ellipsoidal telluroid mapping, which is set up as follows.

A. A. Ardalan, E.W. Grafarend, J. Ihde
Journal of Geodesy (2002) 76: 127-138

Best Fit, Som – Pi Field

Analytically we can formulate the above-stated optimisation problem by minimising the constraint Lagrangean

$$\begin{aligned}\mathbf{L}(x_1,x_2,x_3,x_4) :&= \|\mathbf{x}-\mathbf{X}\|^2 + x_4(W_P - w_p)\\ &= [x - X(x_1,x_2,x_3)]^2 + [y - Y(x_1,x_2,x_3)]^2\\ &\quad + [z - Z(x_1,x_2,x_3)]^2\\ &\quad + x_4[W(x_1,x_2,x_3) - w_p]\\ &= \min_{x_1,x_2,x_3,x_4} \qquad (1)\end{aligned}$$

$$(\hat{x}_1,\hat{x}_2,\hat{x}_3,\hat{x}_4) = \min_{x_1,x_2,x_3,x_4} \mathbf{L}(x_1,x_2,x_3,x_4) \qquad (2)$$

Definition 1. *Somigliana–Pizzetti field as the gravity field of a rotational ellipsoid.*

Explicit form in terms of fundamental geodetic parameters $\{a, b, W_0, \Omega\}$ (according to Grafarend and Ardalan 1999)

$$\begin{aligned}W(\phi,u) = (W_0 - \frac{1}{3}\Omega^2 a^2)\frac{\cot^{-1}(\frac{u}{\varepsilon})}{\cot^{-1}(\frac{b}{\varepsilon})} + \frac{1}{3}\Omega^2(u^2+\varepsilon^2)\\ + \left\{\frac{1}{3\sqrt{5}}\Omega^2 a^2 \frac{(3\frac{u^2}{\varepsilon^2}+1)\cot^{-1}(\frac{u}{\varepsilon}) - 3\frac{u}{\varepsilon}}{(3\frac{b^2}{\varepsilon^2}+1)\cot^{-1}(\frac{b}{\varepsilon}) - 3\frac{b}{\varepsilon}}\right.\\ \left. - \frac{1}{3\sqrt{5}}\Omega^2(u^2+\varepsilon^2)\right\}\frac{\sqrt{5}}{2}(3\sin^2\phi - 1)\end{aligned}$$

Case study: potential quasi-geoid of East Germany

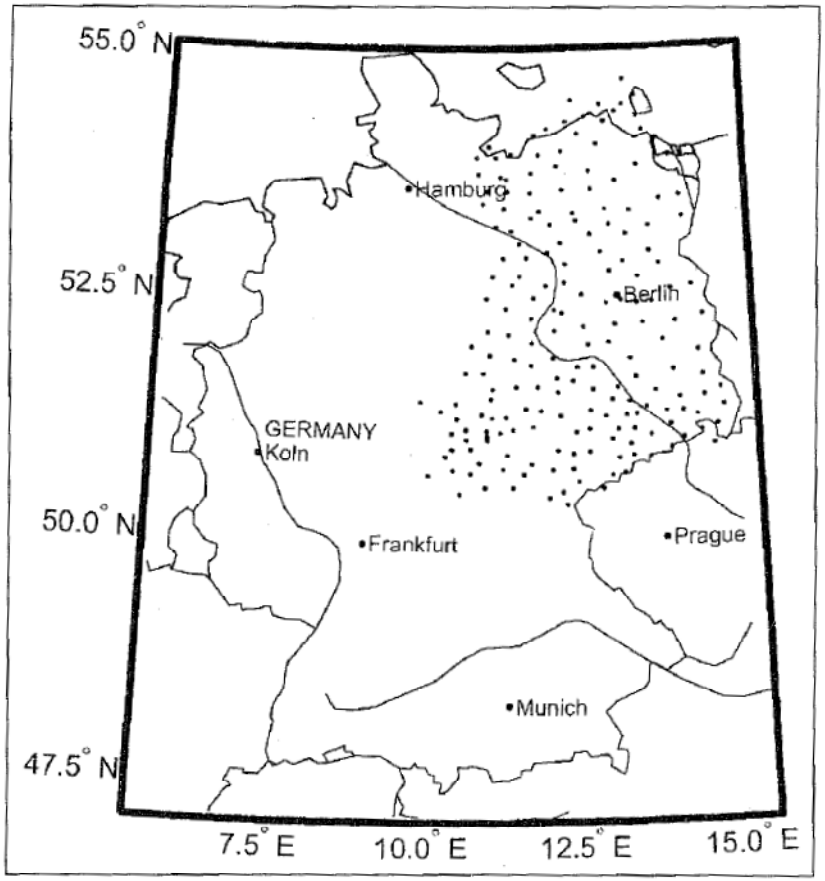


Figure The 196 GPS stations in the eastern part of Germany. Equidistant conic projection; standard parallels: 50°N and 52.5°N; reference ellipsoid: WGD2000

Molodensky potential telluroid

Table 2. Part of the GPS file of the eastern part of Germany

Longitude (l_p) [deg]	Latitude (b_p) [deg]	Ellipsoidal height (h_p) [m]	Geopotential number [m^2/s^2]
13.4363	54.6772	82.295	455.75
13.6433	54.5136	68.192	318.76
12.5016	54.4716	39.663	21.20
13.0076	54.4256	48.669	115.93
13.4252	54.4172	92.697	553.89
13.2909	54.3508	42.444	58.39
12.7371	54.2981	50.352	127.90
13.6586	54.2971	101.363	641.58
13.0796	54.2511	55.762	185.67
12.4053	54.2495	40.304	23.93

Table 3. Transferred Jacobi ellipsoidal coordinates $\{\lambda, \phi, u\}_p$ of the Gauss ellipsoidal coordinates given in Table 2

λ_p	ϕ_p	u_p
13.4363	54.5864	6 356 834.2477
13.6433	54.4226	6 356 820.1294
12.5016	54.3805	6 356 791.5681
13.0076	54.3345	6 356 800.5844
13.4252	54.3261	6 356 844.6627
13.2909	54.2596	6 356 794.3526
12.7371	54.2069	6 356 802.2698
13.6586	54.2059	6 356 853.3393
13.0796	54.1598	6 356 807.6861
12.4053	54.1582	6 356 792.2103

Telluroid Map: East Germany

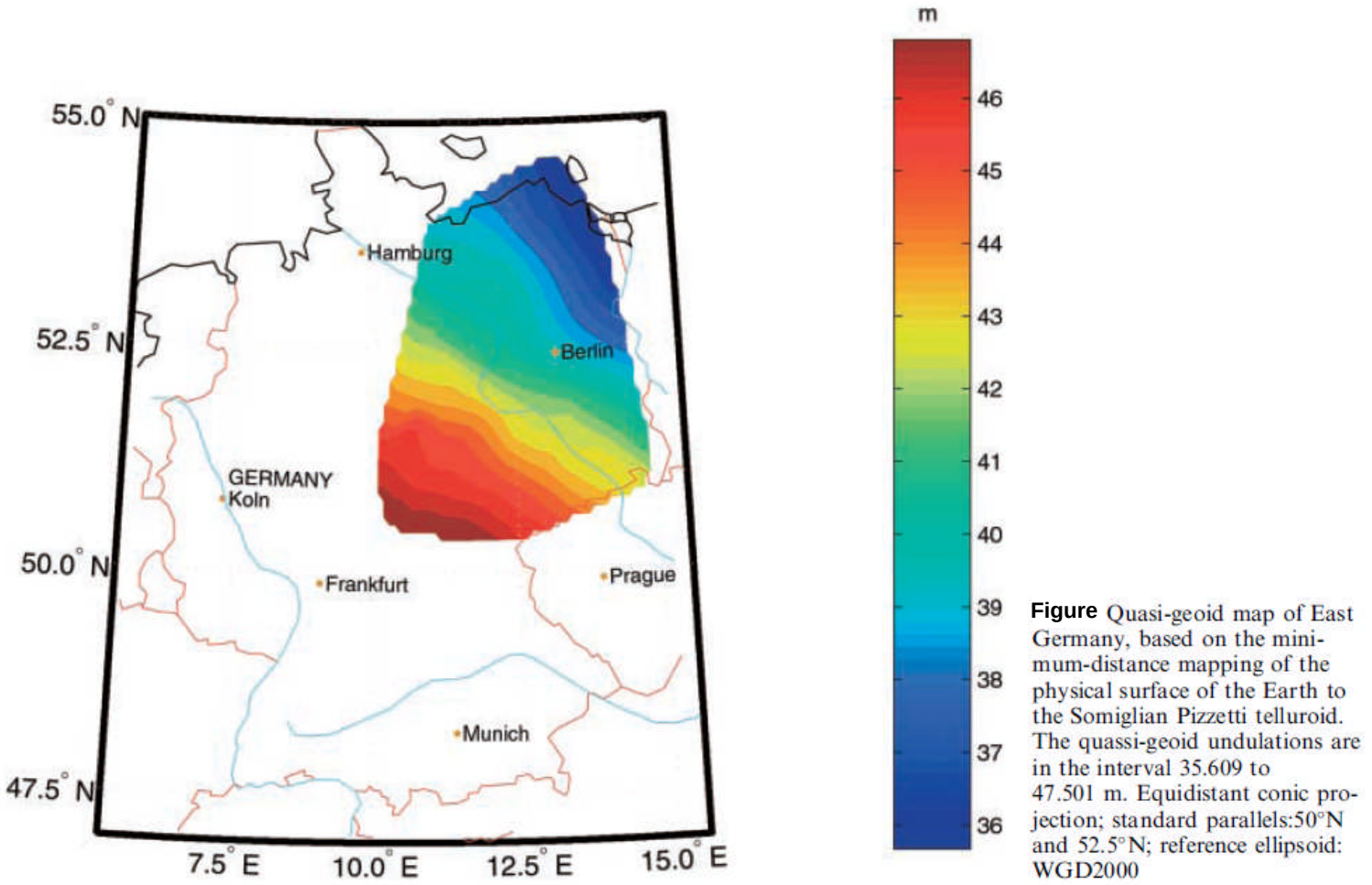
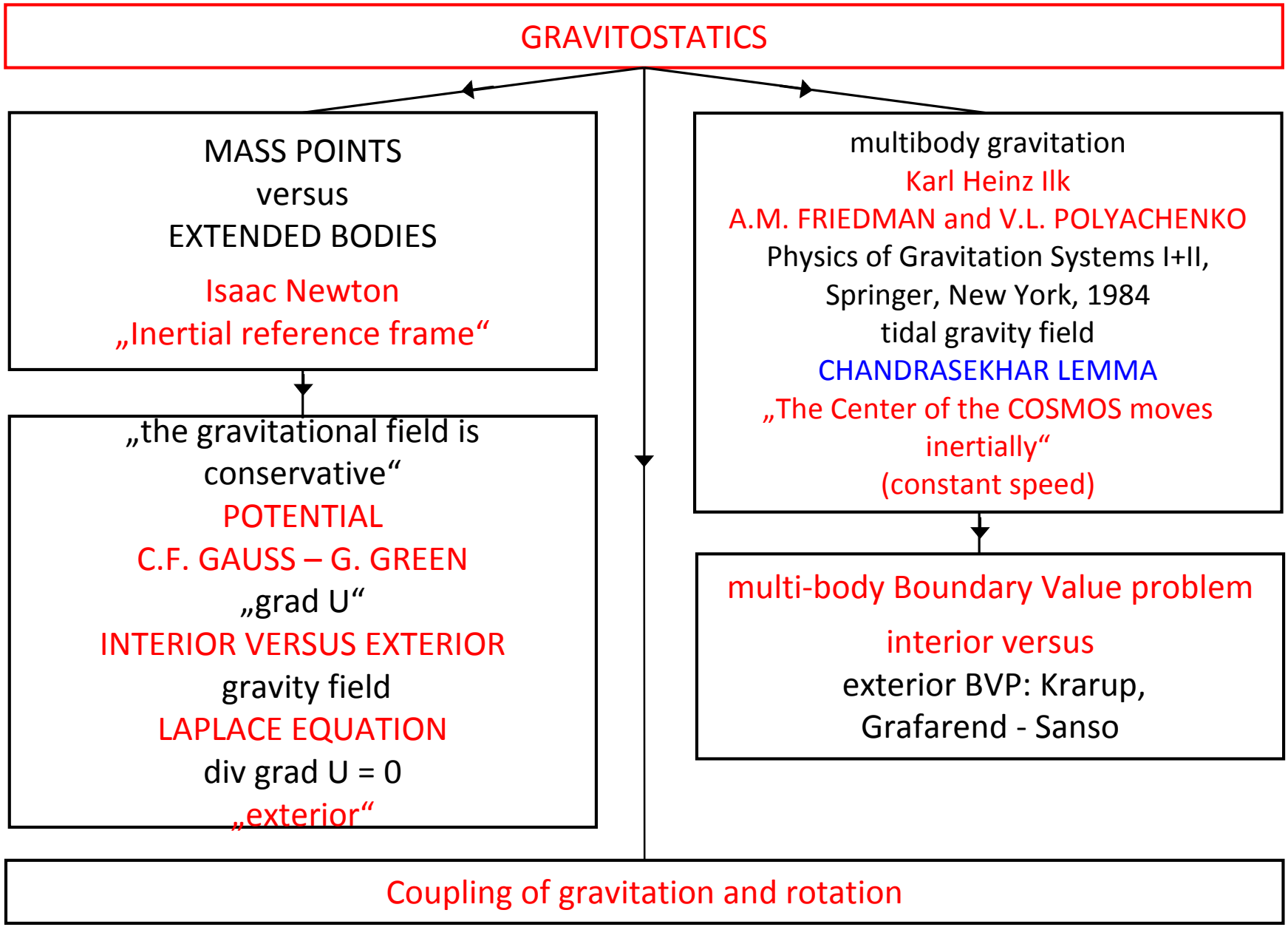


Figure Quasi-geoid map of East Germany, based on the minimum-distance mapping of the physical surface of the Earth to the Somiglian Pizzetti telluroid. The quassi-geoid undulations are in the interval 35.609 to 47.501 m. Equidistant conic projection; standard parallels:50°N and 52.5°N; reference ellipsoid: WGD2000


GRAVITOSTATICS
MASS POINTS
versus
EXTENDED BODIES
Isaac Newton
„Inertial reference frame“
„the gravitational field is conservative“
POTENTIAL
C.F. GAUSS – G. GREEN
„grad U“
INTERIOR VERSUS EXTERIOR
gravity field
LAPLACE EQUATION
div grad U = 0
„exterior“
multibody gravitation
Karl Heinz Ilk
A.M. FRIEDMAN and V.L. POLYACHENKO
Physics of Gravitation Systems I+II,
Springer, New York, 1984
tidal gravity field
CHANDRASEKHAR LEMMA
„The Center of the COSMOS moves inertially“
(constant speed)
multi-body Boundary Value problem
interior versus
exterior BVP: Krarup,
Grafarend - Sanso
Coupling of gravitation and rotation

Coupling of gravitation and rotation

forces:
(i) gravitation, (ii) centrifugal, (iii) EULER

„angular momentum“: length-of-day
polar motion ↔ precession-nutation
„NO POTENTIAL“

(iv) CORIOLIS: currents, flow
„NO POTENTIAL“

HELMHOLTZ decomposition
„grad - rot“

grad(U+V) + rot Γ

(i) div grad W(x) = $2\Omega^2$ („external“)
(ii) rot Γ = $-2\Omega^{\bullet}$

cont. GRAVITOSTATICS

Base functions

spherical harmonics
ellipsoidal harmonics

2-axis → 3-axis
„BIFURCATION“
S. Chandrasekhar

BJERHAMMAR-KRARUP
RUNGE-WALSH Approx. Theorem
ILL-POSED PROBLEMS

IMPROPERLY POSED
DOWNWARD CONTINUATION
BIAS-VERSUS
MEAN SQUARE ERRORS
REGULARISATION

BVP
fixed-free
Grafarend-Ardalan-Sideris
linear-nonlinear
1st, 2nd, 3rd, oblique
mixed

Reference Fields

GEOID-AREOID-PLANETS

TOPOGRAPHY

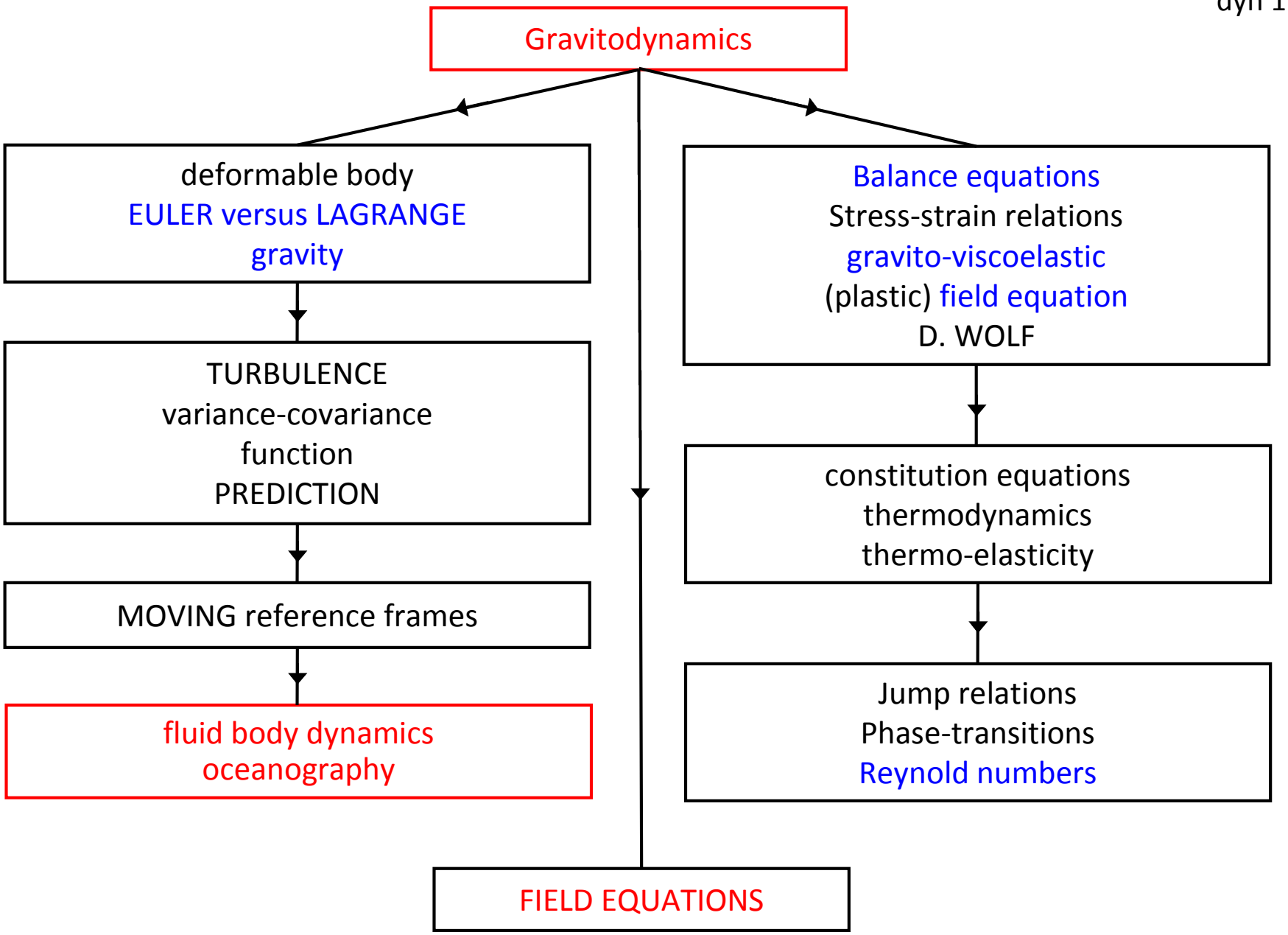
STOKES
BRUNS-VENING-MEINESZ
„generalisations“

PLANE-SPHERE-ELLIPSOID

SOMIGLIANA-PIZZETTI

forces
gravitation, volume forces, surface forces („stress“)
line forces (plasticity, dislocation: Earthquakes)

GRAVITODYNAMICS


Gravitodynamics
deformable body
EULER versus LAGRANGE
gravity
TURBULENCE
variance-covariance
function
PREDICTION
MOVING reference frames
fluid body dynamics
oceanography
Balance equations
Stress-strain relations
gravito-viscoelastic
(plastic) field equation
D. WOLF
constitution equations
thermodynamics
thermo-elasticity
Jump relations
Phase-transitions
Reynold numbers
FIELD EQUATIONS

FIELD EQUATIONS

curl equ.:

$$\operatorname{rot}\boldsymbol{\Gamma} = -2\boldsymbol{\omega}\operatorname{div}\mathbf{x}^{\bullet} + \boldsymbol{\Omega}\operatorname{grad}\langle\mathbf{x}^{\bullet}\,|\,\boldsymbol{\omega}\rangle -$$

$$-2\boldsymbol{\omega}\,\mathbf{x}\operatorname{rot}\mathbf{x}^{\bullet} - 2\boldsymbol{\omega}^{\bullet}$$

divergence equ.:

$$\operatorname{div}\boldsymbol{\Gamma} = 4\pi\rho + 2\omega^2 + 2\langle\boldsymbol{\omega}\,|\operatorname{rot}\mathbf{x}^{\bullet}\rangle$$

VORTICITY rot $\mathbf{x}^{\bullet}$

rigid body

$$\operatorname{rot}\Gamma = -2\omega^{\bullet}$$

$$\operatorname{div}\Gamma = 4\pi g\rho + 2\omega^2$$

hydrostatic prepressed elastic body

$$\boldsymbol{\Sigma} = -p\,\mathbf{I} + c\,\mathbf{E}$$

stress tensor

balance equations

$$\mathrm{d}_{tt}(\rho(\mathbf{x},t)) = \rho\mathbf{I} - \operatorname{grad} p =$$

$$= \rho\operatorname{grad}(\mathrm{U} + \tfrac{1}{2}\,|\,\boldsymbol{\omega}\times\mathbf{x}\,|^2) - \operatorname{grad} p$$

CHANDRASEKHAR'S VIRIAL METHOD

virial equations of order zero, one, two and higher order

EQUILIBRIUM figures of higher order

BIFURCATION

cont. GRAVITODYNAMICS

EQUILIBRIUM FIGURES

Superpotential and
Supermatrix

homogeneous versus heterogeneous
Ellipsoids

THE VIRIAL THEOREM

THEORY OF BIFURCATIONS

? Stability ?

Maclaurin
JACOBI

JEANS
ROCHE
DARWIN

DEDEKIND
RIEMANN