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Fiber bundles and topology for quantum matter

Abstract:

Methods of differential geometry are much used in geodesy and cosmology. Recently they have found entrance in analyzing topologically nontrivial states of quantum matter. An introduction is given into the basic notions.

1. Prologue

Erik Grafarend and I met for the first time in 1985, when I succeeded Prof. Ekkehard Kröner to the Chair for Theoretical and Applied Physics of the University of Stuttgart. In 1966, Erik had written a diploma thesis in physics under the guidance of Kröner at the University of Clausthal. Kröner obtained a call to the University of Stuttgart in 1969, and Erik followed as director of the Institute for Geodesics in Stuttgart in 1980. Both showed very much interest in differential geometric methods, which I shared. At the New Year's reception of the rector in 2014 I reported to Erik most recent applications of differential and topological methods to quantum systems. He got excited and immediately encouraged me to give an overview at the present colloquium. Happy birthday, Erik, and all best wishes!

2. The tangent bundle of the sphere

A notion of differential geometry which gains more and more applications in physics is that of the fiber bundle[1]. Prominent and simple example of a fiber bundle is the tangent bundle of the sphere S^2.

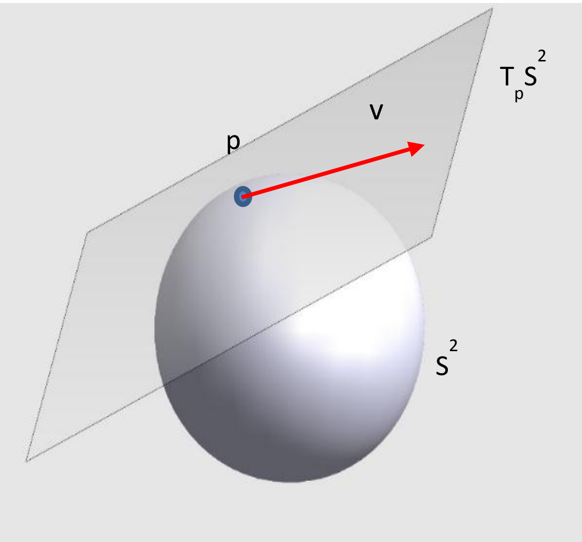

Fig. 1: Sphere and tangent plane

The sphere is so smooth that at each point a tangent plane T_pS^2 can be attached (Fig. 1). T_pS^2 is a vector space. Sphere S^2 plus all tangent planes form the tangent bundle TS^2. S^2 is the basis manifold, the tangent planes are the fibers. A point in TS^2 is characterized by

$$p \in S^2,\ \vec{v} = v_1\vec{e}_1(p) + v_2\vec{e}_2(p) \in T_pS^2$$

where $\{\vec{e}_1(p), \vec{e}_2(p)\}$ is a basis of T_pS^2. There are many choices for bases. For the canonical basis the vectors $\vec{e}_1(p)$ and $\vec{e}_2(p)$ point along the coordinate lines, e.g. the lines of latitude and longitude. For a vector field over the sphere, for example the currents of air or water of the Earth, out of each tangent plane one vector is chosen. It is then denoted "section of the bundle".

Tangent planes at different points p are inclined towards each other. A comparison of vectors out of different tangent planes therefore needs special rules which are given by the notion of "parallel transport along a path". On the sphere, and generally on all two-dimensional surfaces embedded in three-dimensional flat space $\mathbb{R}^3$, two nearby vectors $\vec{u}(p_1)$ and $\vec{v}(p_2)$ are parallel according to Levi-Civita, when $\vec{v}(p_2)$ is carried in $\mathbb{R}^3$ to point p_1 and then its difference to $\vec{u}(p_1)$ points along the surface normal. Given a path $s(t)$ on the surface, a parallel transport operator

$$P(t \to t_0):\quad T_{P(t)} \to T_{P(t_0)},\ \vec{w}(t) \longmapsto \vec{w}(t_0)$$

identifies parallel vectors in tangent planes at $P(t)$ and $P(t_0)$ and allows the definition of a directional derivative for non-parallel vector fields:

$$D_{\vec{u}}\vec{w} = u^\lambda D_\lambda \vec{w} = \lim_{t\to t_0} \frac{1}{t \to t_0}\{P(t \to t_0)\vec{w}(t) - \vec{w}(t_0)\}$$

Here $\vec{u} = \vec{u}(t_0) = \frac{ds}{dt}|_{t_0}$ is then tangent vector to the path $s(t)$ at t_0.

The "covariant derivative" $D_\lambda\vec{w}$ can be expressed as

$$D_\lambda\vec{w} = \vec{e}_\mu\left(\partial_\lambda w^\mu + \Gamma^\mu_{\nu\lambda} w^\nu\right).$$

It contains the usual partial derivative of the vector components. The "connection coefficients" $\Gamma^\mu_{\nu\lambda}$ take regard of the fact that the basis vectors of the tangent planes are in general not parallel. The index λ denotes the direction of the derivative in the basis manifold, indices μ and ν act as in a matrix on the vector components. A vector field $\vec{w}$ is parallel along $s(t)$, if $D_{\vec{u}}\vec{w} = \vec{0}$.

3. Curvature

Curvature of a manifold becomes evident, when parallel transport occurs along a closed path (Fig. 2). Then the parallel transported vector differs from the original one. On the sphere the difference consists of a rotation by the angle

$$\Delta\omega = \int_{\text{encircled area}} d\Omega\, \kappa(\vartheta, \varphi)$$

which is the integral of the Gaussian curvature over the encircled area. If one is intersecting a surface with a plane containing the surface normal vector, it cuts out a curve to which a circle of radius R can be adapted. Of all such planes there are always two orthogonal ones with maximal and minimal curvature radii $R_1$and R_2, and the Gaussian curvature is defined as $\kappa = \frac{1}{R_1}\frac{1}{R_2}$.

The Gaussian curvature is the only quantity entering the curvature tensor of a two-dimensional surface. This tensor, which will not be explained in more detail here, can be calculated from the connection coefficients $\Gamma^\varrho_{\lambda\nu}$ by an antisymmetrized derivative

$$R^\varrho_{\lambda\mu\nu} = \partial_\mu\Gamma^\varrho_{\lambda\nu} - \partial_\nu\Gamma^\rho_{\lambda\mu}$$

which resembles a curl. The indices μ and ν refer to derivatives along directions in the basis manifold, the indices ϱ and λ again act like matrix indices on tangent vectors. The only non-zero components of the curvature tensor for a two-dimensional surface are

$$R^1_{212} = -R^1_{221} = -R^2_{112} = R^2_{2121} = \kappa$$

where 1 and 2 denote normalized coordinate basis vectors.

Parallel transport according to Levi-Civita preserves the lengths of the vectors, hence merely causes rotations. Thus, for the description of parallel transport and holonomy, it suffices to attach to each point of the manifold not an entire tangent plane, but only the rotation group $SO(2)$ acting on it. For the sphere this results in the "principle bundle" $\left(S^2, SO(2)\right) = (S^2, S^1)$, where S^1 is the circle isomorphic to $SO(2)$. One has to fix the position of the unit element of $SO(2)$ on each circle. A change is possible and results in a "gauge transformation" of the connection coefficients $\Gamma^{\varrho}_{\lambda\nu}$.

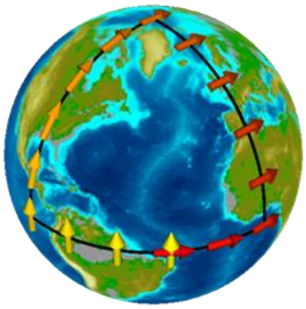

Fig. 2 Parallel transport along a closed loop

4. Fiber bundles

Now we are ready to define a fiber bundle (E, M, F, π) in general. It consists of a basis manifold, to which at each point a fiber F_p is attached. The fiber is either of a copy of a vector space V ("vector bundle") or a (gauge) group G acting on V ("principal bundle"). $\pi: E \to M$ is a projection such that the fiber F_p is the inverse image $\pi^{-1}(p)$ of a point $p \in M$.

An example is the Moebius strip with the circle S^1 as basis manifold and the real line $\mathbb{R}$ as vector fiber (Fig. 3). In addition prescriptions must be given for gluing the fibers together to create a twist.

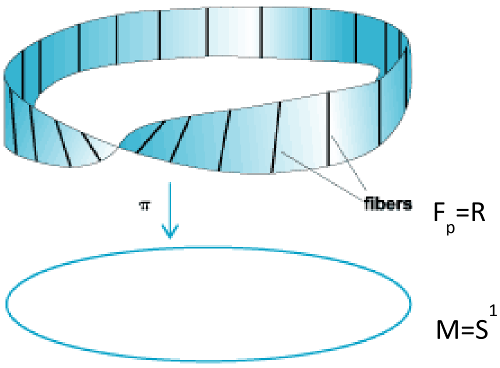

Fig. 3 Moebius strip

For comparison of vectors in different fibers also a parallel transport can be defined with covariant derivative

$$D_\lambda \vec{w} = \vec{e}_i \left(\partial_\lambda w^i + \Gamma^i_{j\lambda} w^j\right).$$

Greek indices now denote directions or tangent vectors in the basis manifold, Latin indices vector components in the fiber. If we write the vectors as columns of components $\boldsymbol{w} = \begin{bmatrix} w^1 \\ w^2 \end{bmatrix}$, the covariant derivative can be expressed as matrix equation

$$\boldsymbol{D}_\lambda \boldsymbol{w} = (\partial_\lambda + \boldsymbol{\Gamma}_\lambda)\boldsymbol{w},$$

with the connection coefficient matrix $[\boldsymbol{\Gamma}_\lambda]^i_j = \Gamma^i_{j\lambda}$.
The curvature tensor follows as

$$R^i_{j\mu\nu} = \partial_\mu \Gamma^i_{j\nu} - \partial_\nu \Gamma^i_{j\mu} + \Gamma^i_{l\mu}\Gamma^l_{j\nu} - \Gamma^i_{l\nu}\Gamma^l_{j\mu}$$

$$\boldsymbol{R}_{\mu\nu} = \partial_\mu \boldsymbol{\Gamma}_\nu - \partial_\nu \boldsymbol{\Gamma}_\mu + [\boldsymbol{\Gamma}_\nu, \boldsymbol{\Gamma}_\mu]$$

The terms in red are commutators that exist only for nonabelian gauge groups.

5. Topological quantum numbers

The theorem of Gauss-Bonnet states that the integral

$$\chi_E(M) = \frac{1}{2\pi}\int_M d\Omega\, \kappa(\vartheta, \varphi) \;= 2 - 2g$$

over the sphere or any closed two-dimensional surface is an integer, namely the Euler characteristic. g is the genus of the surface, the number of handles. An example for $\chi_E(M) = -4$ is given in Fig. 4.

Fig. 4 Surface of genus 3 and Euler characteristic -4

A continuous deformation of the surface cannot change the Euler characteristic, as it is a discrete index. Thus the Euler characteristic serves to label those classes of surfaces which can be continuously transformed into each other and constitutes a topological quantum number. The Gauss-Bonnet theorem connects differential geometry with topology. The topological method to determine the Euler characteristic is by covering the surface with polygons. Then $\chi_E(M)$ is the number of faces plus the number of vertices minus the number of edges (Fig. 5 for the sphere).

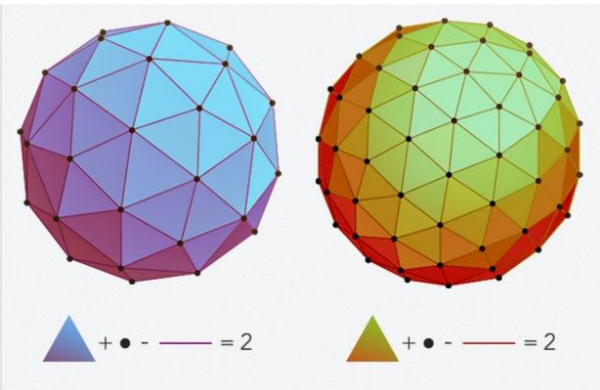

Fig. 5 Determination of the Euler characteristic by triangulation

This method is more general as it requires only a continuous manifold and not a differentiable one with parallel transport. Before we come to topological quantum numbers in quantum matter, we address some other applications of fiber bundles.

6. Applications of fiber bundles

The tangent bundle TM^4 of space-time with principle bundle $\left(M^4, SO(1,3)\right)$ is used to describe the cosmos. The tangent planes are Lorentz spaces upon which the Lorentz group $SO(1,3)$ is acting.

In special relativity one defines the canonical four-momentum

$$p^\mu = \begin{bmatrix} E/c \\ \vec{p} \end{bmatrix}$$

where E is the energy, c the velocity of light, and $\vec{p}$ the standard momentum. The four-velocity is $v^\mu = p^\mu/m$. In an electromagnetic field, four-velocity and canonical four-momentum are related by

$$v^\mu = \frac{1}{m}(p^\mu - qA^\mu)$$

where q is the charge and

$$A^\mu = \begin{bmatrix} \Phi \\ \vec{A} \end{bmatrix}$$

the four-potential with Φ the scalar and $\vec{A}$ the vector potential. In quantum mechanics a particle is described by a complex wave function $\psi(\vec{r}, t) \in \mathbb{C}$, i.e. a section of a bundle $(\mathbb{R} \otimes \mathbb{R}^3, \mathbb{C})$ with space-time $\mathbb{R} \otimes \mathbb{R}^3$ as basis manifold and the $\mathbb{C}$ as fiber. The canonical four-momentum and four-velocity become differential operators

$$p_\mu \to \frac{\hbar}{i}\partial_\mu \quad \text{and} \quad v_\mu \to \frac{\hbar}{im}D_\mu,\ D_\mu \equiv \partial_\mu - i\frac{q}{\hbar}A_\mu.$$

$\hbar$ is the Planck-constant divided by 2π and i the imaginary unit. D_μ serves as a covariant derivative and defines parallelism. The factor $q/\hbar$ is as a rule absorbed in A_μ. The true wave function along a path is a parallel transported one, hence the solution of $D_{\vec{u}}\psi = 0$, which is

$$\psi = \psi_0 \exp\{i \int dx^\mu A_\mu\}.$$

The exponent is purely imaginary and therefore a change of the phase. Parallel transport is connected with the action of the rotation group (or circle) $U(1)$ in the complex plane; we have to deal with a$\left(\mathbb{R} \otimes \mathbb{R}^3, U(1)\right)$ principle bundle. Whereas the connection matrix is $-iA_\mu$ (acting on the complex plane), the curvature is $-iF_{\mu\nu}$ with the electromagnetic field tensor

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$

in Cartesian components.

Similar to the Euler characteristic for a closed surface – which had been an integral over curvature – a topological quantum number can be associated with the electromagnetic field on a closed surface, also by an integral over the generalized curvature, denoted first Chern number

$$ch^{(1)} = \frac{1}{2\pi} \int_{\substack{\text{closed} \\ \text{submanifold}}} dx^1 dx^2\, F_{12}$$

x^1 and x^2 denote the coordinates on the two-dimensional surface. This is a generalization of the Gauss-Bonnet theorem to fiber bundles and called Gauss-Bonnet-Chern theorem.

An example is a magnetic monopole of strength γ. It can be described as $(S^2, U(1))$ fiber bundle with linear connection and curvature:

$$-iA_\vartheta = 0, \qquad -iA_\varphi = \gamma i(\cos\vartheta - 1), \quad -iF_{\vartheta\varphi} = -\gamma i \sin\vartheta$$

This is the Levi-Civita connection transcribed for the complex plane $\mathbb{C}$. Compare it with the connection and curvature matrices for S^2 in an orthogonal basis:

$$\boldsymbol{\Gamma}_\vartheta = 0, \qquad \boldsymbol{\Gamma}_\varphi = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{i}(\cos\vartheta - 1), \qquad \boldsymbol{R}_{\vartheta\varphi} = -\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{i}\sin\vartheta$$

The matrix corresponds to the 90° rotation in $\mathbb{R}^2$which in $\mathbb{C}$ is performed by the imaginary unit. The first Chern number is

$$ch^{(1)} = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta\, F_{\vartheta\varphi} = 2\gamma$$

The $(\mathbb{R} \otimes \mathbb{R}^3, U(1))$ fiber bundle or gauge theory for electromagnetism has been generalized to nonabelian gauge theories $(\mathbb{R} \otimes \mathbb{R}^3, U(1) \otimes SU(2))$ for the electroweak interaction and $(\mathbb{R} \otimes \mathbb{R}^3, SU(3))$ for the strong interaction.

7. Topological quantum states

Linear connection and curvature were introduced in quantum mechanics with the discovery of the Berry phase in adiabatic motions. We consider quantum mechanical ground states $|m(\xi)\rangle$ which depend on a parameter ξ out of a parameter manifold M. An example is a particle with spin $1/2$ in a magnetic field of fixed strength but varying orientation. For this case $\xi = (\vartheta, \varphi)$ are the polar angles, and M is the sphere S^2. Quantum mechanical states are determined up to a phase $e^{i\alpha}$, therefore the system can be described as an $(M, U(1))$ principle bundle. Assume that the states are separated from adjacent ones by an energy gap and that the parameter is changed adiabatically so slow, that the applied energy does not suffice to surmount the gap, then the system remains in the instanta-neous states $|m(\xi)\rangle$. However, upon completing a motion along a closed path in M the phase will have changed. This holonomy can be described by a covariant derivative $D_\mu \equiv \partial_\mu - iA_\mu$ containing the

Berry connection $\quad A_\mu = i\langle m|\partial_\mu m\rangle \quad$ and

Berry curvature $\quad F_{\mu\nu} = i\{\langle\partial_\mu m|\partial_\nu m\rangle - \langle\partial_\nu m|\partial_\mu m\rangle\}.$

The derivatives are taken with respect to the parameter ξ.

Given a curvature, a first Chern number can be calculated for a two-dimensional parameter manifold as above. For the spin-$1/2$ particle we obtain the same connection coefficients and curvature as for the magnetic monopole, only γ being replaced by $1/2$.

8. The quantum Hall effect

The first topological quantum state was discovered by von Klitzing in 1980 in the quantum Hall effect[2]. There, a crystalline semiconductor layer forming a two-dimensional electron gas is placed into a perpendicular magnetic field B_z (Fig. 6).

The charge carriers of a current j_x are deflected by the magnetic field and cause a transversal electric field E_y. Current and field are related by the Hall conductivity σ_{xy}: $j_x = \sigma_{xy}E_y$. The Hall conductivity turned out to be

$$\sigma_{xy} = \frac{e^2}{h}n$$

where e is the charge of the electron, h Planck's constant and n an integer which changed with the strength of the magnetic field.

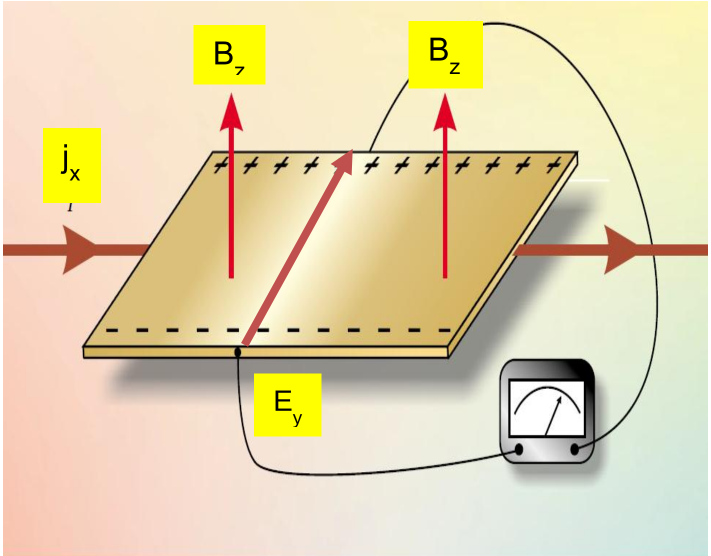

Fig. 6 Configuration for the quantum Hall effect

Two years later the quantized nature of the Hall conductivity was explained in a famous paper by Thouless, Kohmoto, Nightingale and den Nijs[3]. The system is periodic in two directions, therefore the quantum states $|u(\vec{k})\rangle$ are labeled by a wave vector $\vec{k}$=(k_1, k_2). The wave vector space is also periodic: $\vec{k} \in \mathbb{R}^2 \mathrm{mod}\, (\vec{K}_1, \vec{K}_2)$, and thus forms a torus T^2(Fig. 7).

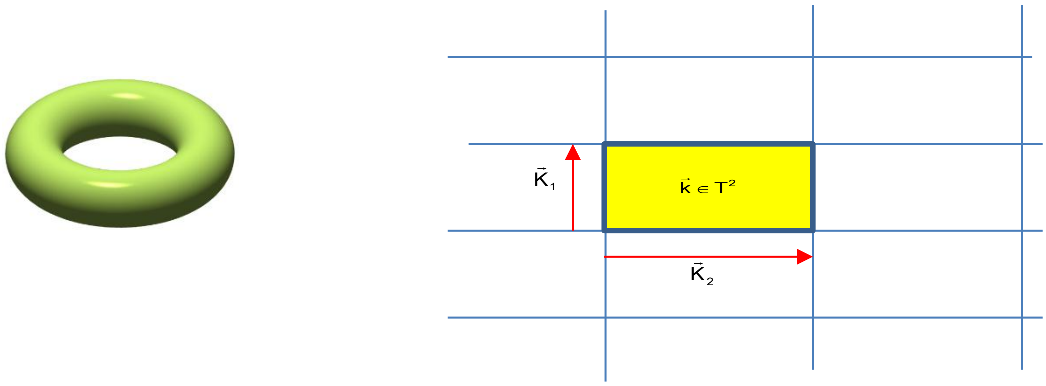

Fig. 7 A periodic wave vector space is equivalent to a torus

The Kubo transport equation for the Hall conductivity is

$$\sigma_{xy} = \frac{e^2}{h} \sum_{\text{occupied bands}} \frac{1}{2\pi} \int_{\text{torus}} d^2\,\vec{k}\; i\{\langle \partial_1 u | \partial_2 u \rangle - \langle \partial_2 u | \partial_1 u \rangle\}$$

The expression in the curly brace is exactly the Berry curvature, and hence the integers in the Hall conductivity are a sum of Chern numbers:

$$\sigma_{xy} = \frac{e^2}{h} \sum_{\text{occupied bands}} \mathrm{ch}^{(1)}\left(T^2, U(1)\right)$$

9. The spin quantum Hall effect

Continuous deformations may perturb spatial symmetries, for example cubic symmetry of a crystal. But there are extremely generic symmetries that may persist, for example time-reversal or particle hole symmetries. These impose restrictions on the topological quantum numbers. Time reversal symmetry transforms a state at wavevector $\vec{k}$ to one at wavevector $-\vec{k}$, hence on the torus these points must be identified. This action modifies the basis manifold and requires new methods for the determination of topological quantum numbers, like twisted K-theory.

An example is the spin quantum Hall effect[4]. It occurs in isolators with large spin-orbit coupling. The role of the magnetic field is taken by the electron spin. The classification is not any more by integers $\mathbb{Z}$ but by $\mathbb{Z}_2 = \mathbb{Z} \bmod 2$. It has been predicted and observed in two-dimensional HgTe/CdTe quantum well structures and three-dimensional $Bi_{1-x}Sb_x$ crystals. The systems are isolators in the bulk but show exotic metallic spin polarized currents on the boundaries.

10. Summary

Up to 1980 quantum numbers were based on symmetry only, denoting irreducible represenations. Symmetries are easy to break, which causes lifting of degeneracies. The explanation of the quantum Hall effect, which was discovered in 1980, required the introduction of topological quantum numbers. These describe new physical phenomena, like exotic electronic states on the surfaces of topological isolators. They are invariant towards deformations of the band structure and hence extremely robust. They are realized by Chern numbers of fiber bundles with abstract curvatures.

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