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Summation of series and an approximation of Legendre functions in constructing integral kernels for the exterior of an ellipsoid: Application to boundary value problems in physical geodesy

Abstract:

Integral kernels are an important tool used in solving boundary value problems. The paper primarily concerns physical geodesy applications and thus problems associated with Laplace's and Poisson's partial differential equation that offer a natural basis for gravity field studies. In the introduction a brief review is given on Green's function constructed for Stokes' and Neumann's problem formulated for the exterior of a sphere. The second of the problems is considered also within the weak solution concept. Galerkin elements are expressed for the special case when the function basis is generated by the respective reproducing kernel or represented by reciprocal distances (elementary potentials). The solution domain is then generalized and the paper focuses on the construction of the reproducing kernel of Hilbert's space of functions harmonic in the exterior of an oblate ellipsoid of revolution. In the first stage the kernel is represented by a series of ellipsoidal harmonics. However, the manipulation with the series and a numerical implementation of the integral kernel is rather demanding in this case, in particular for a high resolution modeling of the solution. This stimulates studies towards an analytical summation of the series representing the kernel. In solving this problem some approximations of Legendre functions of the second kind are used. Two alternatives are discussed. The first approach is based on the solution of an approximation version of Legendre's ordinary differential equation (limit layer approach). In the second alternative the Legendre functions are approximately represented by hypergeometric functions and series. In both the cases the integral kernel is split into parts given by series of Legendre functions of the first kind. The use of Legendre elliptic integrals for their summation is pointed out. The results based on the above two alternatives applied for the approximation representation of the Legendre functions of the second kind are discussed.

1. Introduction

In geodesy we are dealing with a number of interesting boundary value problems of mathematical physics. We usually work in a system of Cartesian coordinates x_1, x_2, x_3 such that its origin is in the center of gravity of the Earth and its x_3 axes coincides with the rotation axes of the Earth. We follow a classical or weak solution concept and often use an integral representation of the solution, Green's function method or also Galerkin approximations of the solution. Traditionally and very often solution methods applied in this area rest on a mathematical apparatus that actually was developed for a spherical boundary.

A classical example is the famous Stokes function $G(\boldsymbol{x},\boldsymbol{y})$. It is associated with the problem to find a function T such that

$$\frac{\partial T}{\partial |\boldsymbol{x}|}+\frac{2}{R}T=f \quad \text{for } |\boldsymbol{x}|=R \quad \text{and} \quad \Delta T=g \quad \text{for } |\boldsymbol{x}|>R \ , \tag{1}$$

where $R > 0$ is a constant, $|\boldsymbol{x}| = (x_1^2 + x_2^2 + x_3^2)$ and Δ means Laplace's differential operator. Apart from the first degree harmonics $T_1(\boldsymbol{x})$ the function $G(\boldsymbol{x}, \boldsymbol{y})$ enables to write the solution of the problem in the following form

$$T(\boldsymbol{x}) = T_1(\boldsymbol{x}) - \frac{1}{4\pi} \int_{|\boldsymbol{y}|=R} G(\boldsymbol{x}, \boldsymbol{y})\, f(\boldsymbol{y})\, d_y S - \frac{1}{4\pi} \int_{|\boldsymbol{y}|>R} G(\boldsymbol{x}, \boldsymbol{y}) g(\boldsymbol{y})\, d\boldsymbol{y} \ , \tag{2}$$

where $d_y S$ is an element of surface at the point $\boldsymbol{y}$ and $d\boldsymbol{y} = dy_1 dy_2 dy_3$ means the volume element. For the exterior of the sphere of radius R Stokes' function can even be expressed in a closed form. We have

$$\begin{aligned} G(\boldsymbol{x}, \boldsymbol{y}) = {} & \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{R}{|\boldsymbol{x}|} \frac{1}{|\bar{\boldsymbol{x}}-\boldsymbol{y}|} - 3\frac{R|\bar{\boldsymbol{x}}-\boldsymbol{y}|}{|\boldsymbol{x}||\boldsymbol{y}|^2} \\ & - \frac{R^3 \cos\psi}{|\boldsymbol{x}|^2 |\boldsymbol{y}|^2} \left[5 + 3\ln\frac{1}{2}\left(1 - \frac{R^2\cos\psi}{|\boldsymbol{x}||\boldsymbol{y}|} + \frac{|\bar{\boldsymbol{x}}-\boldsymbol{y}|}{|\boldsymbol{y}|} \right) \right], \end{aligned} \tag{3}$$

where $\bar{\boldsymbol{x}} = (R^2 / |\boldsymbol{x}|^2)\, \boldsymbol{x}$ and ψ is the angle between the position vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, see (Holota, 1985, 1995, 2003b, 2014).

Remark 1. $G(\boldsymbol{x}, \boldsymbol{y})$ above is a more general in comparison with Stokes' function usually applied in physical geodesy, see e.g. (Heiskanen and Moritz, 1967), (Moritz, 1980) and (Hofmann-Wellenhof and Moritz, 2005). It enables to solve the geodetic boundary value problem not only for Laplace's equation but also for Poissons' partial differential equation.

Another example is Neumann's function $N(\boldsymbol{x}, \boldsymbol{y})$. It is associated with the problem to find T such that

$$\frac{\partial T}{\partial |\boldsymbol{x}|} = f \ \text{ for } |\boldsymbol{x}| = R \quad \text{and} \quad \Delta T = g \ \text{ for } |\boldsymbol{x}| > R \ . \tag{4}$$

Equally as above we also have a closed form expression of this function

$$N(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{R}{|\boldsymbol{x}|} \frac{1}{|\bar{\boldsymbol{x}}-\boldsymbol{y}|} - \frac{1}{R} \ln \frac{|\bar{\boldsymbol{x}}-\boldsymbol{y}| + |\bar{\boldsymbol{x}}| - |\boldsymbol{y}|\cos\psi}{|\boldsymbol{y}|(1-\cos\psi)} \ , \tag{5}$$

see (Holota, 2003b, 2004). In physical geodesy now its restrictions for $|\boldsymbol{y}| = R$ or for $|\boldsymbol{x}| = |\boldsymbol{y}| = R$ are often applied. In particular for $|\boldsymbol{y}| = R$, it is obvious that $N = N(\boldsymbol{x}, \psi)$ and we easily deduce that

$$N = \frac{2}{|\boldsymbol{x}-\boldsymbol{y}|} - \frac{1}{R} \ln \frac{|\boldsymbol{x}-\boldsymbol{y}| + R - |\boldsymbol{x}|\cos\psi}{|\boldsymbol{x}|(1-\cos\psi)} \ . \tag{6}$$

cf. (Pick et al., 1973) or (Holota, 2003b). In the case that $|\boldsymbol{x}| = |\boldsymbol{y}| = R$, we even have

$$N = N(\psi) = \frac{1}{R} \left[\frac{1}{\sin(\psi/2)} - \ln \frac{1 + \sin(\psi/2)}{\sin(\psi/2)} \right] \ , \tag{7}$$

see (Hotine, 1969), (Pick et al.,1973), (Holota, 2003b) and (Hoffmann-Wellenhof and Moritz, 2005).

In geodesy kernel functions have their position also within the concept of the weak solution of boundary value problems. Good example is the reproducing kernel. Nevertheless, its construction corresponds not only to the geometry of the solution domain, but also to the scalar product associated with the respective Hilbert space of functions.

Consider e.g. Hilbert's space $H_2^{(1)}(S_R)$ of functions harmonic in the exterior of the sphere of radius R which is equipped with the following scalar product $(u,v)_1 = A(u,v)$, where

$$A(u,v) = \sum_{i=1}^{3} \int_{S_R} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, d\boldsymbol{x} \ . \tag{8}$$

One can show that there exists a function $K(\boldsymbol{x},\boldsymbol{y})$ such that

$$\sum_{i=1}^{3} \int_{S_R} \frac{\partial K(\boldsymbol{x},\boldsymbol{y})}{\partial x_i} \frac{\partial v(\boldsymbol{x})}{\partial x_i} \, d\boldsymbol{x} = v(\boldsymbol{y}) \quad \text{holds for all} \quad v \in H_2^{(1)}(S_R) \tag{9}$$

and that in this case

$$K(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi R} \sum_{n=0}^{\infty} \frac{2n+1}{n+1} z^{n+1} P_n(\cos\psi) \ , \quad z = \frac{R^2}{|\boldsymbol{x}||\boldsymbol{y}|} \ , \tag{10}$$

where P_n is the usual Legendre polynomial of degree n, see (Holota, 2004). Moreover, it is not extremely difficult to find that

$$K(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi R} \left(\frac{2z}{L} - \ln \frac{L + z - \cos\psi}{1 - \cos\psi} \right) , \tag{11}$$

where $L = \sqrt{1 - 2z\cos\psi + z^2}$, see also (Tscherning, 1975), (Neyman, 1979) and (Holota, 2004). The advantage of using the kernel function $K(\boldsymbol{x},\boldsymbol{y})$ can be seen from the construction of Galerkin approximations to the solution of Neumann's problem for Laplace's equation.

Recall, therefore, that in this case we are looking for u such that

$$\Delta u = 0 \ \text{ in } \ S_R \quad \text{and} \quad \frac{\partial u}{\partial n} = -f \ \text{ on } \ \partial S_R \ , \tag{12}$$

where $\partial / \partial n$ denotes the derivative in the direction of the unit (outer) normal $\boldsymbol{n}$ of ∂S_R.

In the numerical solution the function u is approximated by the linear combination

$$u_n = \sum_{j=1}^{n} c_j^{(n)} v_j \ , \tag{13}$$

where v_j are members of a function basis of $H_2^{(1)}(S_R)$. The coefficients $c_j^{(n)}$ can be obtained from the respective Galerkin system

$$\sum_{j=1}^{n} c_j^{(n)} A(v_j, v_k) = \int_{\partial S_R} v_k \, f \, dS \ , \quad k = 1, \ldots, n \ , \tag{14}$$

where

$$A(v_j, v_k) = \sum_{i=1}^{3} \int_{S_R} \frac{\partial v_j}{\partial x_i} \frac{\partial v_k}{\partial x_i} \, d\boldsymbol{x} \ , \tag{15}$$

see (Nečas, 1967), (Michlin, 1964), (Rektorys, 1974), (Holota, 1997, 2004, 2005, 2011). Putting now

$$v_j(\boldsymbol{x}) = K(\boldsymbol{x}, \boldsymbol{y}_j) \ , \tag{16}$$

we can immediately deduce that in our Galerkin system the elements $A(v_j, v_k)$ are given by

$$A(v_j, v_k) = K(\boldsymbol{y}_j, \boldsymbol{y}_k) \ , \tag{17}$$

which results from the reproducing property of the kernel, see (Holota, 2004).

The advantage of using the function $K(\boldsymbol{x}, \boldsymbol{y})$ definitely attracts attention. Nevertheless, in constructing Galerkin approximations it is tempting to generate basis functions also in a different way. Often elementary potentials are used for this purpose, which means that we put

$$v_j(\boldsymbol{x}) = \frac{1}{|\boldsymbol{x} - \boldsymbol{y}_j|} \ . \tag{18}$$

In this case one can show that the diagonal elements are given by

$$A(v_j, v_j) = \pi \left(\frac{2R}{R^2 - |\boldsymbol{y}_j|^2} + \frac{1}{|\boldsymbol{y}_j|} \ln \frac{R + |\boldsymbol{y}_j|}{R - |\boldsymbol{y}_j|} \right) \ , \tag{19}$$

see (Holota, 2000). Concerning the off diagonal elements, however, the computation is more laborious. In particular, we have to use a series representation of $v_j(\boldsymbol{x})$, i.e.,

$$v_j(\boldsymbol{x}) = \sum_{n=0}^{\infty} \frac{|\boldsymbol{y}_j|^n}{|\boldsymbol{x}|^{n+1}} P_n(\cos\psi_j) \ , \tag{20}$$

where ψ_j is the angle between the position vectors $\boldsymbol{x}$ and $\boldsymbol{y}_j$, see e.g. (Heiskanen and Moritz, 1967) or (Hofmann-Wellenhof and Moritz, 2005). This enables us to get

$$A(v_j, v_k) = \frac{2\pi}{R} \sum_{n=0}^{\infty} z^n P_n(\cos\psi_{jk}) + \frac{4\pi}{R} S \ , \tag{21}$$

where ψ_{jk} is the angle between the position vectors $\boldsymbol{y}_j$ and $\boldsymbol{y}_k$,

$$z = \frac{|\boldsymbol{y}_j||\boldsymbol{y}_k|}{R^2} \tag{22}$$

and

$$S = \sum_{n=0}^{\infty} \frac{1}{2n+1} z^n P_n(\cos\psi_{jk}) \ , \tag{23}$$

Here, however, we are faced by the problem to sum the series on the right hand side of Eq. (23) or find a closed expression for the off diagonal element $A(v_j, v_k)$. What we get is

$$A(v_j, v_k) = \frac{2\pi}{R\sqrt{1 - 2z\cos\psi_{jk} + z^2}} + \frac{4\pi}{R} S \ , \tag{24}$$

where

$$S = \frac{1}{\sqrt{z}} \int \frac{1}{\sqrt{z\left(1-2z\cos\psi_{jk}+z^2\right)}} dz \tag{25}$$

is an elliptic integral, see (Holota, 1999) and (Holota and Nesvadba, 2014).

2. Transition to an ellipsoid of revolution

In the introduction we saw that the mathematical apparatus related to the exterior of the sphere is fairly developed and transparent. Seemingly, this has an advantage. However, a sphere is rather far the shape of the Earth, which has a negative effect on any attempt to reach the solution of the boundary value problem formulated for the real surface of the Earth by means of an iteration procedure. Therefore, our aim is to discuss the construction of the mathematical apparatus for the exterior Ω_{ell} of an oblate ellipsoid of revolution. This leads us to an attempt to construct an analogue to $K(\boldsymbol{x},\boldsymbol{y})$, i.e. the reproducing kernel $K_{ell}(\boldsymbol{x},\boldsymbol{y})$ for Hilbert's space $H_2^{(1)}(\Omega_{ell})$ of functions harmonic in Ω_{ell} which is equipped with scalar product $(u,v)_1 = A(u,v)$, where

$$A(u,v) = \sum_{i=1}^{3} \int_{\Omega_{ell}} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, d\boldsymbol{x} \ . \tag{26}$$

A note motivating interest in ellipsoidal reproducing kernels can be found also in (Tscherning, 2004).

We will suppose that a and b, $a \geq b$, are the semi-axes of the oblate ellipsoid of revolution and will consider the ellipsoidal coordinates u, β, λ related to x_1, x_2, x_3 by the equations

$$x_1 = \sqrt{u^2+E^2}\cos\beta\cos\lambda \ , \quad x_2 = \sqrt{u^2+E^2}\cos\beta\sin\lambda \ , \quad x_3 = u\sin\beta \ , \tag{27}$$

where $E = \sqrt{a^2-b^2}$ denotes the linear eccentricity, see e.g. Heiskanen and Moritz (1967). Clearly, the boundary $\partial\Omega_{ell}$ is then defined by $u = b$.

Going now back to $K_{ell}(\boldsymbol{x},\boldsymbol{y})$, and referring to (Holota, 2004, 2011) and (Holota and Nesvadba, 2014), we can write immediately that

$$\begin{aligned} K_{ell}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi b}\sum_{n=0}^{\infty}(2n+1)\Big[K_{n0xy} P_n(\sin\beta_x) P_n(\sin\beta_y) + \\ + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!} K_{nmxy}\, P_{nm}(\sin\beta_x) P_{nm}(\sin\beta_y)\cos m(\lambda_x-\lambda_y)\Big] \end{aligned} \tag{28}$$

with

$$K_{nmxy} = \frac{iEb}{a^2}\frac{Q_{nm}(z_x)}{Q_{nm}(z_0)}\frac{Q_{nm}(z_y)}{Q_{nm}(z_0)}\left[\frac{1}{Q_{nm}(z_0)}\frac{dQ_{nm}(z_0)}{dz}\right]^{-1} , \tag{29}$$

where P_{nm} and Q_{nm} are associated Legendre functions of the first and the second kind, respectively; while

$$z_x = \frac{iu_x}{E} \ , \quad z_y = \frac{iu_y}{E} \ , \quad z_0 = \frac{ib}{E} \quad \text{and} \quad i = \sqrt{-1} \ . \tag{30}$$

The problem, however, is the summation of the series that represents the kernel $K_{ell}(\mathbf{x},\mathbf{y})$. Nevertheless, inspecting the structure of the coefficients K_{nmxy}, we can find a stimulus to use the limit layer theory as discussed in (Sona,1995) or (Sansò and Sona, 2001) and also analyzed in Holota (2003c).

3. Approximation - limit layer theory

As it is known

$$v_{nm}(u) = \frac{Q_{nm}(z)}{Q_{nm}(z_0)} = Q_{nm}\left(i\frac{u}{E}\right)Q_{nm}^{-1}\left(i\frac{b}{E}\right) \tag{31}$$

is a solution of

$$(u^2+E^2)\frac{d^2v_{nm}}{du^2}+2u\frac{dv_{nm}}{du}-\left[n(n+1)-\frac{E^2m^2}{u^2+E^2}\right]v_{nm}=0 \ , \tag{32}$$

which can be shown to be equivalent to Legendre's equation by using the pure imaginary variable $z = iu/E$. This equation has two solutions: Legendre's function of the first kind P_{nm} and Legendre's function of the second kind Q_{nm}. Recall that in our case when dealing when with harmonic functions in an unbounded solution domain the function Q_{nm} is the suitable solution.

As in (Sona, 1995), in the sequel we will work with the variable $s = u/b$. It is thus obvious that $v_{nm}(u) = v_{nm}(sb) = v(s)$ and that for the function v Legendre's equation transforms into

$$(s^2+e^2)\frac{d^2v}{ds^2}+2s\frac{dv}{ds}-\left[n(n+1)-\frac{e^2m^2}{s^2+e^2}\right]v=0 \ , \tag{33}$$

where $e^2 = E^2/a^2$. Now, confining ourselves to $s \in \langle 1, s_{\max}\rangle$, where $s_{\max}$ is an upper bound for s, such that

$$e^2s^{-2} \approx e^2 \quad \text{and} \quad (s^2+e^2)^{-1} \approx (1+e^2)^{-1} \tag{34}$$

can be taken for admissible approximations, we can simplify the equation above. It transforms into

$$(1+e^2)s^2\frac{d^2w}{ds^2}+2s\frac{dw}{ds}-\left[n(n+1)-\frac{e^2m^2}{1+e^2}\right]w=0 \ . \tag{35}$$

The solution $w(s)$ obviously differs from $v(s)$. Following (Sona, 1995) again, we have

$$w(s) = \frac{A}{s^\alpha} \ , \tag{36}$$

where A is a constant and α has to be one of the two roots, $\alpha_1 \approx -n$ and $\alpha_2 \approx n+1$, of the quadratic equation

$$(1+e^2)\alpha^2-(1-e^2)\alpha-n(n+1)+\frac{e^2m^2}{1+e^2}=0 \ . \tag{37}$$

Clearly, α_2 is the suitable solution in our case and we obtain

$$w(s)=\frac{1}{s^{n+1-\varepsilon}}\ ,\qquad \varepsilon=e^2\,\frac{(n+1)(n+2)+m^2}{2n+1}\ , \tag{38}$$

where A has been fixed so that $w(1)=1$. Note in particular that $v(s)=w(s)=1$ for $s=1$.

As regards the differenced of the two solutions (i.e. v and w) it has been estimated by means of Gronwall's inequality in Holota (2003c).

4. Coefficients in the expansion of $K_{ell}(\boldsymbol{x},\boldsymbol{y})$

Within the limit layer theory, we can write approximately that

$$\frac{Q_{nm}(z_x)}{Q_{nm}(z_0)}\approx w\left(\frac{u_x}{b}\right)=\left(\frac{b}{u_x}\right)^{n+1-\varepsilon}\ ,\quad \frac{Q_{nm}(z_y)}{Q_{nm}(z_0)}\approx w\left(\frac{u_y}{b}\right)=\left(\frac{b}{u_y}\right)^{n+1-\varepsilon} \tag{39}$$

and subsequently derive that

$$\frac{1}{Q_{nm}(z_0)}\,\frac{dQ_{nm}(z_0)}{dz}\ \approx\ \frac{E}{ib}\,\frac{dw(1)}{ds}=\frac{iE}{b}\,(n+1-\varepsilon)\ , \tag{40}$$

which holds, but again with some accuracy only, see (Holota, 2011).

Nevertheless, keeping the accuracy up to terms multiplied by e^2, we can substantially modify the right hand side of Eq. (40). We obtain

$$\frac{1}{Q_{nm}(z_0)}\,\frac{dQ_{nm}(z_0)}{dz}\ \approx\ i\,\frac{Eb}{a^2}(n+1)\left[1+\frac{E^2}{ab}\,\frac{(n+1)(n-1)-m^2}{(n+1)(2n+1)}\right]\ . \tag{41}$$

Thus in $K_{ell}(\boldsymbol{x},\boldsymbol{y})$ given by Eq. (28) we will have

$$K_{nmxy}\ \approx\ \left(\frac{b^2}{u_x u_y}\right)^{n+1-\varepsilon}\kappa_{nm}\approx\left(\frac{b^2}{u_x u_y}\right)^{n+1}\left[1-\varepsilon\,\ln\frac{b^2}{u_x u_y}\right]\kappa_{nm}\,, \tag{42}$$

see e.g. (Čuřík, 1944), with

$$\kappa_{nm}=\ \frac{1}{n+1}\left[1-\frac{E^2}{ab}\,\frac{(n+1)(n-1)-m^2}{(n+1)(2n+1)}\right]\ . \tag{43}$$

Moreover, denoting by ψ the angular distance of points (β_x,λ_y) and (β_y,λ_y) on a sphere, when β and λ are interpreted as spherical latitude and longitude, respectively and using the well-known Legendre addition theorem, we can deduce that

$$\begin{aligned}K_{ell}(\boldsymbol{x},\boldsymbol{y})\approx{}&\frac{1}{4\pi b}K^{(1)}(\boldsymbol{x},\boldsymbol{y})-\frac{E^2}{4\pi ab^2}K^{(2)}(\boldsymbol{x},\boldsymbol{y})+\frac{E^2}{4\pi ab^2}K^{(3)}(\boldsymbol{x},\boldsymbol{y})-\\&-\frac{E^2}{4\pi a^2 b}\left(\ln\frac{b^2}{u_x u_y}\right)K^{(4)}(\boldsymbol{x},\boldsymbol{y})+\frac{E^2}{4\pi a^2 b}\left(\ln\frac{b^2}{u_x u_y}\right)K^{(5)}(\boldsymbol{x},\boldsymbol{y})\end{aligned} \tag{44}$$

with

$$K^{(1)}(\boldsymbol{x},\boldsymbol{y})=\sum_{n=0}^{\infty}\frac{2n+1}{n+1}\left(\frac{b^2}{u_x u_y}\right)^{n+1}P_n(\cos\psi)\ , \tag{45}$$

$$K^{(2)}(\boldsymbol{x},\boldsymbol{y})=\sum_{n=0}^{\infty}\frac{n-1}{n+1}\left(\frac{b^2}{u_x u_y}\right)^{n+1}P_n(\cos\psi)\ , \tag{46}$$

$$K^{(3)}(\boldsymbol{x},\boldsymbol{y})=-\sum_{n=1}^{\infty}\frac{1}{(n+1)^2}\left(\frac{b^2}{u_x u_y}\right)^{n+1}\frac{\partial^2 P_n(\cos\psi)}{\partial^2\lambda}\ , \tag{47}$$

$$K^{(4)}(\boldsymbol{x},\boldsymbol{y})=\sum_{n=0}^{\infty}(n+2)\left(\frac{b^2}{u_x u_y}\right)^{n+1}P_n(\cos\psi) \tag{48}$$

and

$$K^{(5)}(\boldsymbol{x},\boldsymbol{y})=\sum_{n=1}^{\infty}\frac{1}{n+1}\left(\frac{b^2}{u_x u_y}\right)^{n+1}\frac{\partial^2 P_n(\cos\psi)}{\partial^2\lambda}\ . \tag{49}$$

where λ means λ_x or λ_y. The terms containing the derivative of P_n with respect to λ resulted from the dependence of κ_{nm} on the index m and from the use of Legendre's adition theorem twice differentiated with respect to λ_x or λ_y in treating the summation of the series on the right hand side of Eq. (28), cf. (Holota, 2003a).

5. Coefficients in $K_{ell}(x,y)$ - an alternative concept of approximation

In our approach we so far used an exact solution of an approximate differential equation. Let us try now to change this philosophy and work with an approximate solution of an exact differential equation. In this case we first recall that

$$Q_{nm}(z)\ =\ (-1)^m\frac{2^n n!(n+m)!}{(2n+1)!}(z^2-1)^{-\frac{n+1}{2}}F\left(\frac{n+m+1}{2},\frac{n-m+1}{2},\frac{2n+3}{2};\frac{1}{1-z^2}\right)\ , \tag{50}$$

where F is a hypergeometric function, see (Bateman and Erdélyi, 1953) or (Hobson, 1931) and that, passing to hypergeometric series,

$$F_z=F\left(a,b,c;\frac{1}{1-z^2}\right)=1+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n\,n!}\left(\frac{1}{1-z^2}\right)^n \tag{51}$$

with $(a)_n=a(a+1)\ldots(a+n-1),\ n=1,2,3\ldots$ [similarly for $(b)_n$ and $(c)_n$]. After elementary modifications we have

$$F_z=F_{z_0}+\frac{ab}{c}\left(\frac{1}{1-z^2}-\frac{1}{1-z_0^2}\right)+\frac{a(a+1)b(b+1)}{2c(c+1)}\left[\left(\frac{1}{1-z^2}\right)^2-\left(\frac{1}{1-z_0^2}\right)^2\right]+\cdots\ , \tag{52}$$

see (Holota and Nesvadba, 2014). Thus, e.g., in a $20km$ layer close above the ellipsoid

$$\frac{1}{1-z^2}-\frac{1}{1-z_0^2} \le e^4 = 0,000046 \tag{53}$$

and

$$\left(\frac{1}{1-z^2}\right)^2-\left(\frac{1}{1-z_0^2}\right)^2 \le 2e^6 = 0,0000006 \tag{54}$$

etc. In the sequel, therefore, considering these estimates, we can approximately put

$$\frac{Q_{nm}(z_x)}{Q_{nm}(z_0)}\frac{Q_{nm}(z_y)}{Q_{nm}(z_0)} \approx \rho^{n+1} \ , \tag{55}$$

where

$$\rho = \frac{1-z_0^2}{\sqrt{1-z_x^2}\sqrt{1-z_y^2}} = \frac{a^2}{\sqrt{u_x^2+E^2}\sqrt{u_y^2+E^2}} \ , \tag{56}$$

see (Holota and Nesvadba, 2014). Similarly recalling that

$$\frac{dQ_{nm}}{dz} = \frac{(n+1)\,z}{1-z^2}Q_{nm} - \frac{n-m+1}{1-z^2}Q_{n+1,m} \ , \tag{57}$$

see (Kratzer and Franz, 1960) and putting

$$\frac{Q_{n+1,m}(z_0)}{Q_{nm}(z_0)} \approx \frac{n+m+1}{2n+3}\frac{E}{ia} \ , \tag{58}$$

see (Holota, 2011), we arrive at

$$\frac{1}{Q_{nm}(z_0)}\frac{dQ_{nm}(z_0)}{dz} \approx i\frac{Eb}{a^2}(n+1)\left[1+\frac{E^2}{ab}\frac{(n+1)^2-m^2}{(n+1)(2n+3)}\right] . \tag{59}$$

Hence

$$K_{nmxy} \approx \rho^{n+1}\kappa_{nm} \quad \text{with} \quad \kappa_{nm} = \frac{1}{n+1}\left[1-\frac{E^2}{ab}\frac{(n+1)^2-m^2}{(n+1)(2n+3)}\right] , \tag{60}$$

see (Holota and Nesvadba, 2014).

Now we return to the reproducing kernel $K_{ell}(\boldsymbol{x},\boldsymbol{y})$. Denoting again by ψ the angular distance of points (β_x,λ_y) and (β_y,λ_y) on a sphere, when β and λ are interpreted as spherical latitude and longitude, respectively, we can deduce that

$$K_{ell}(\boldsymbol{x},\boldsymbol{y}) \approx \frac{1}{4\pi b}K^{(1)}(\boldsymbol{x},\boldsymbol{y}) - \frac{E^2}{4\pi ab^2}K^{(2)}(\boldsymbol{x},\boldsymbol{y}) + \frac{E^2}{4\pi ab^2}K^{(3)}(\boldsymbol{x},\boldsymbol{y}) \tag{61}$$

with

$$K^{(1)}(\boldsymbol{x},\boldsymbol{y})=\sum_{n=0}^{\infty}\frac{2n+1}{n+1}\rho^{n+1}P_n(\cos\psi) \ , \tag{62}$$

$$K^{(2)}(\boldsymbol{x},\boldsymbol{y})=\sum_{n=0}^{\infty}\frac{2n+1}{2n+3}\rho^{n+1}P_n(\cos\psi) \tag{63}$$

and

$$K^{(3)}(\boldsymbol{x},\boldsymbol{y})=-\sum_{n=1}^{\infty}\frac{2n+1}{(n+1)^2(2n+3)}\rho^{n+1}\frac{\partial^2 P_n(\cos\psi)}{\partial^2\lambda} \ , \tag{64}$$

where λ means λ_x or λ_y. Note also that similarly as in the last section Legendre's addition theorem and its second derivative with respect to λ have been used in deriving the terms $K^{(1)}(\boldsymbol{x},\boldsymbol{y})$, $K^{(2)}(\boldsymbol{x},\boldsymbol{y})$ and $K^{(3)}(\boldsymbol{x},\boldsymbol{y})$, see (Holota and Nesvadba, 2014).

6. Concluding remarks

The concepts applied for the approximation computation of the coefficients K_{nmxy} in Sections 4 and 5 differs. Details concerning the summation of the series in Eqs (62) - (64) are discussed in (Holota and Nesvadba, 2014) and also in (Holota, 2014). Therefore, we will confine ourselves just to the results. The right hand side of Eq. (62) has the same structure as the series in Eq. (10). Here the summation of $K^{(1)}(\boldsymbol{x},\boldsymbol{y})$ is not a problem. However, for the $K^{(2)}(\boldsymbol{x},\boldsymbol{y})$ and $K^{(3)}(\boldsymbol{x},\boldsymbol{y})$ term the summation of the series in Eqs (63) and (64) is somewhat more complex. In particular the derivation of closed formulas for $K^{(2)}(\boldsymbol{x},\boldsymbol{y})$ and $K^{(3)}(\boldsymbol{x},\boldsymbol{y})$ involves the use of Legendre elliptic integrals of the first and the second kind, see (Holota and Nesvadba, 2014) and (Holota, 2014). In addition in (Holota and Nesvadba, 2014) the difference between $K_{ell}(\boldsymbol{x},\boldsymbol{y})$ given by Eq. (28) and its approximation represented by Eqs (61) - (64) was subjected extensive numerical tests.

In contrast the results obtained in Section 4 and especially the structure of the series in Eqs (46) - (49) have a somewhat different nature. As already mentioned in Section 4 the difference between the function $v(s)$ and its approximation $w(s)$ based on the limit layer theory was estimated by means of Gronwall's inequality in (Holota, 2003c). This confirmed that in Ω_{ell} the approximation is acceptable in a close neighborhood of the boundary $\partial\Omega_{ell}$. Note, however, that the question concerning a similar estimate valid for the difference between the first derivatives of these two functions was (though considered) not fully answered so far.

Nevertheless, it is worth mentioning that in Section 4 the analytical summation leading to the closed form representation of $K^{(n)}(\boldsymbol{x},\boldsymbol{y})$, $n=2,\dots,5$, does not need the use of special tools. One can show that the series in Eqs (46) - (49) are summable without the use of elliptic integrals. This is an interesting finding and definitely also a stimulus for research. The problem will be further discussed in the next paper on this topic.

Finally, note that there exists a rather extensive literature dealing with the implementation of the global flattening of the Earth into the mathematical apparatus used for the solution of geodetic boundary value problems in gravity field studies. An excerpt from this literature can be found in (Holota and Nesvadba, 2014). It contains also numerous references to works by Prof. Erik W. Grafarend to whom this contribution is dedicated.

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