Leibniz Online, Jahrgang 2015, Nr. 19
Zeitschrift der Leibniz-Sozietät e. V.
ISSN 1863-3285
http://www.leibnizsozietaet.de/wp-content/uploads/2015/06/burghardt.pdf

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Collapsing gravity models considered critically.

1. Preliminary remarks

Many studies have been published in recent decades to ensure the phenomenon of the star collapse as part of Einstein's gravitational theory. The problem arose that Einstein's differential equation system is generally not sufficient to determine the necessary variables which describe the collapse. There are only a few approaches which are analytical solutions of Einstein's field equations and of which the integration constants correspond to physical quantities. With respect to physical requirements these approaches are very simplified. They describe stars consisting of pressureless dust. These are the models of Oppenheimer and Snyder and the ones of McVittie and Weinberg. Although these models have been well known for decades and widely quoted, they offer some surprising properties after a closer examination. We want to present some of them in summary form.

2. Pressureless models

McVittie [1] has treated a class of collapsing solutions which are intended to describe stellar objects which consist of incoherent dust with homogeneous density and are pressure-free. Since in this case the inner resistance against contraction is missing, the object cannot be static. It collapses due to its own gravitational forces.

A completely pressure-free star is not physically realistic. During a collapse the particles of a star finally come so close that at a sufficiently high density pressure can be expected. However, a pressure-free stellar object can approximately describe a dying star. If the thermonuclear processes are exhausted inside a star, the star gives way to its own gravitational attraction and collapses. The simplification to freedom of pressure just discussed has good practical reasons. The integration of Einstein's field equations without this condition leads to considerable difficulties, a useful analytical solution is hard to find.

The class of McVittie consists of three models, described by the line element

$$ds^2 = \mathcal{K}^2 \left[\frac{1}{1 - k\frac{r'^2}{\mathcal{R}^2}} dr'^2 + r'^2 d\Omega^2 \right] - dt'^2 \; . \qquad (2.1)$$

Therein k is a parameter which can take on the values $k = \{1, 0\, {-1}\}$. $\mathcal{K} = \mathcal{K}(t')$ is the *scale factor*, describing the time course of the contraction of the star and $d\Omega^2 = d\vartheta^2 + \sin^2\vartheta\, d\varphi^2$ the lateral part of the line element. $\{r', \vartheta, \varphi, t'\}$ are the comoving coordinates, the coordinates of an observer following the collapse. Later on, we will use the non-comoving coordinates $\{r, \vartheta, \varphi, t\}$. $\mathcal{R}_0$ is a constant which will have a geometric interpretation. Requirement for any non-rotating collapsing star should be that it is surrounded by a Schwarzschild field, the *linking condition* must be satisfied.

Since we want to describe the problems in a simple way as possible, we make use of the option to treat the models geometrically, ie to represent them as surfaces in a higher-dimensional flat embedding space. Therefore it is immediately clear that the case with $k = -1$ is a space with an imaginary curvature radius and cannot be connected to the Schwarzschild model with real curvature

The case $k=-1$ has not been further treated in the literature. $k=0$ corresponds to the model of Oppenheimer and Snyder [2] and $k=1$ was rediscovered by Weinberg [3] at a later time and was discussed by him in more detail. The two models have been studied by us in numerous papers and the results have been clearly summarized in [4]. In the following Sections we go into some peculiarities of these models.

3. The model of Oppenheimer and Snyder

In 1939, Oppenheimer and Snyder presented a paper that is now known as that paper that gave rise to the theory of black holes, although the term 'black hole' was introduced much later. In addition, the approach of OS differs from today's methods to implement black holes. The OS model consists of a collapsing interior and an exterior solution, whereby the exterior part is the static Schwarzschild exterior solution which remains static due to the Birkhoff theorem, even if the field generating stellar object collapses.

The OS model is based on an existing cosmological solution of Tolman [4], where OS have taken the hardly insightful notation of Tolman, which is brought by us into the common form of the literature

(3.1) $$(A)\quad ds^2 = \mathcal{K}^2\left[dr'^2 + r'^2 d\Omega^2\right] - dt'^2 .$$

The relation between the radial comoving coordinate r' and the non-comoving r is

(3.2) $$r = \mathcal{K} r', \quad \mathcal{K} = \mathcal{K}(t').$$

OS have explicitly given the quantity $\mathcal{K}$. Since $\mathcal{K}$ does not depend on r', one has for the coordinate transformation between the two systems $\partial r/\partial r' = \mathcal{K}$. In addition, OS have specified the relation between t' and t. Thus, a matrix Λ can be compiled for the transformation between the two coordinate systems. Obviously, it is quite tedious to discover such a transformation. Probably for this reason, other authors did not specify such a coordinate transformation for their models, or the putting up of such a coordinate transformation has not been possible because the model does not have an analytical solution. Above all, the use of different coordinate systems has the only purpose to make calculations as simple as possible. The main advantage lies in the fact that such a coordinate transformation is accompanied by a Lorentz transformation which includes velocity parameters. If one has found such a Lorentz transformation, one has on hand the physical velocity of the collapse as well, if one refers to the velocity at the surface of the stellar object.

From (3.2) and the complicated relation between t and t ', which can be found in the paper of OS, we can calculate by redifferentiation the holonomic, but rather complicated transformation matrix

$$\Lambda^i_{k'} = x^i_{|k'}$$

and thus, we are able to write down the line element in non-comoving coordinates

(3.3) $$(B)\quad ds^2 = \alpha^2 dr^2 + r^2 d\Omega^2 + a_T^2 dit^2, \quad \alpha = \frac{1}{\sqrt{1-\frac{r^2}{\mathcal{R}^2}}} .$$

Due to the similarity with static models, in particular with the Schwarzschild interior solution, the OS model can be interpreted geometrically. The spatial part of (3.3) is the line element on a spherical cap of radius $\mathcal{R}$. For the polar angle η of the spherical cap one has

(3.4) $$r = \mathcal{R}\sin\eta, \quad \alpha = 1/\cos\eta .$$

Thus, the tools for embedding into a higher-dimensional space are provided. η is simultaneously the angle of ascent of the tangent planes of the spherical cap and

(3.5) $$v = -\sin\eta = -\frac{r}{\mathcal{R}}$$

is the velocity of the matter particles inside the star. As usual, we interpret the geometry of the spatial part of the exterior solution as Flamm's paraboloid. The radial curvature radius of it is easy to calculate and is twice the radius of the curvature of the spherical cap at the boundary surface of the two geometries. Thus

(3.6) $$\mathcal{R} = \sqrt{\frac{r_g^3}{2M}}$$

applies if is r_g the radial coordinate at the boundary surface. Inserting (3.6) into (3.5) we obtain the Schwarzschild values

(3.7) $$v_g = -\sqrt{\frac{2M}{r_g}}, \quad \alpha_g = \frac{1}{\sqrt{1-\frac{2M}{r_g}}} \,.$$

The complicated assembled time-like metrical factor a_T of OS, which we did not explicitly write down, matches the Schwarzschild value at the boundary surface as well. We have discussed details in our paper [5].

This result is based on the assumption that the interior OS geometric model is based on a spherical cap. We will directly derive these relations from the details provided by OS. Since one can read from the line elements (A) and (B) the 4-bein system one can construct a matrix

$$L_n^{m'} = \overset{m'}{e}_{i'} \Lambda_k^{i'} \underset{n}{e}^k$$

which has the simple structure

(3.8) $$L_1^{1'} = \alpha, \quad L_4^{1'} = i\alpha v, \quad L_1^{4'} = -i\alpha v, \quad L_4^{4'} = \alpha$$

but whose components α and v are still quite intricate. We recognize the transformation as a pseudo rotation in the local [1,4]-tangential planes. Physically interpreted, it is a Lorentz transformation which makes the connection between a static reference system and a comoving frame, both related to the interior of the star.

If one has calculated the expressions (3.8) for the surface of the collapsing star, one has gained the physically relevant value for the collapse velocity of the star and the associated Lorentz factor. This again gives the values (3.7). Thus, v_g is the velocity of an object coming in free fall from *infinity*, and has reached the boundary surface of the geometries. This means that the surface of the star itself was at infinity at the beginning of the collapse. The star was infinitely large and collapses in free fall, whereby it is surrounded by a Schwarzschild field.

We would have obtained the same result if we had examined the exterior solution of OS. By a simple gauge transformation of the radial coordinate one obtains from the OS-line element the well-known line element for the free fall in the Schwarzschild field found by Lemaître [5].

We want to deepen the problem. We translate the expression given by OS for the mass density of the star with the help of the relations used by us and thus we obtain

(3.9) $$\kappa\mu_0 = \frac{3}{\mathcal{R}^2} \,,$$

the standard expression for the mass density, which is known from other models, in particular from the interior Schwarzschild solution. Having derived the form of $\mathcal{R}$ from the geometry with (3.6), one gets from (3.9)

(3.10) $$\kappa\mu_0 = \frac{6M}{r_g^3} \cdot$$

For $r_g \to \infty$, thus at the beginning of the collapse, the mass density is zero. The space which fills the star is infinitely large, but empty.

It is assumed that the star can shrink to a point. According to (3.10) the mass density would be infinitely large at this location. The surface of the star is said to have crossed the event horizon. According to (3.6) there is $\mathcal{R} = 2M$. The cap of the sphere then is a hemisphere and adjoins the circle at the waist of Flamm's paraboloid. As it is evident from (3.7) the collapse velocity would have reached the velocity of light. A further shrinking of the star would destroy the geometric picture. All the known problems of the event horizon of Schwarzschild geometry will occur.

The solution of Oppenheimer and Snyder is an analytical solution of Einstein's field equations, which equally describes the interior and the exterior of a stellar object. It fulfills all the formal requirements of a field theory based on the laws of general relativity. The above considerations suggest, however, that the surface of the object can only asymptotically approach the event horizon. The formation of a black hole is not possible from this point of view. We refer to the paper of Mitra [6] and his theory of ECOs and MECOs. Mitra [7] has shown that OS have made a mistake by the less significant factor of $1/4$ at the critical part of the respective calculation. In addition, the crucial variable has the wrong sign for penetrating the event horizon.

The physical usability of the model has been questioned. A pressure-free star which fills the vastness of the universe and has vanishing density at the beginning of the collapse and leaves an empty space with a Schwarzschild field during the collapse behind it, does not exist.

4. The model of Weinberg

In his textbook Weinberg presents a model for a collapsing star. Demanding spherical symmetry and choosing suitable initial conditions Weinberg obtains a collapsing interior solution which he adapts somewhat intricately to the exterior Schwarzschild solution. Since the solution is already known to us, we can write the metric in a form that allows us to represent the model clearly arranged:

(4.1) $$\text{(A)}\quad ds^2 = K^2\left[\frac{1}{1-\frac{r'^2}{\mathcal{R}_0^2}}dr'^2 + r'^2 d\vartheta^2 + r'^2\sin^2\vartheta\, d\varphi^2\right] - dt'^2 \cdot$$

In this model the surface of the star is located at the beginning of the collapse at a finite distance from the center of gravity. Formally, the process of shrinkage is formulated with the help of two velocities

(4.2) $$v_R = -\frac{r}{\mathcal{R}}, \quad v_I = -\frac{r'}{\mathcal{R}_0}$$

and the associated Lorentz factors α_R and α_I. Again the relation (3.2) is valid. As with the OS model $\mathcal{R} = \mathcal{R}(t')$ can be interpreted as the radius of a spherical cap. $\mathcal{R}_0$ is the initial value of $\mathcal{R}$ and is a constant. One finds the relation $\mathcal{R} = \sqrt{K^3}\mathcal{R}_0$.

(4.3) $$\text{(B)}\quad ds^2 = \alpha_R^2 dr^2 + r^2 d\Omega^2 - a_T^2 dt^2, \quad \alpha_R = 1\Big/\sqrt{1 - r^2/\mathcal{R}^2}$$

is the line element in the non-comoving system. Weinberg notes a complicated expression for a_T which takes the well-known form of the surrounding Schwarzschild field on the surface of the star.

$\partial K/\partial t'$ can be determined from the field equations. That is enough to set up together with (3.2) two necessary components of the transformation matrix Λ. By transvecting this matrix with the bein vectors of both systems one determines the Lorentz transformation which converts one reference system into one another. From it one reads the velocity of the particles inside the stellar object and the associated Lorentz factor

(4.4) $$v_C = -\alpha_I\sqrt{v_R^2 - v_I^2}, \quad \alpha_C = \frac{\alpha_R}{\alpha_I}.$$

Weinberg allows the star to shrink under the event horizon and to contract to a point. This needs further consideration. A light signal which is emitted radially from the surface of the collapsing star needs an infinitely long coordinate time to reach an observer if the surface coincides with the event horizon. This observer sees the star to collapsing eternally, but below the event horizon the star cannot be observed by him. Due to the increasing redshift he sees the star fade slowly when the surface of the star approaches the event horizon.

We consider this scenario described by Weinberg hardly realistic. On the one hand the star should collapse to a point with infinitely high mass density in a relatively short proper time, on the other hand this star should have a finite extent for an external observer for all time. In accordance with the principle of relativity, it is evident that two observers which are in different states of motion, have also different views about the chronological sequence. But the principle of relativity does not mean that an object can have two different mutually excluding states. We therefore assume an inconsistency in the model of Weinberg and we want to investigate this problem.

For the line element in the Schwarzschild field one has

(4.5) $$ds^2 = dl^2 - dT^2 = dl'^2 - dT'^2 = dl''^2 - dT''^2,$$

whereby the expressions stand for a non-comoving system, a system coming in free fall from a finite position r_0 and a system coming in free fall from infinity. T, T' and T'' are the associated proper times. The 4-dimensional distance ds between two adjacent points is an invariant, it has the same value for all three observers. This, of course, is not true for the 3-dimensional distances and the time intervals. For the three observers is $ds \neq 0$ and $ds = 0$ is basically excluded due to the invariance of ds. $ds = 0$ stands for null-lines, ie for observers who would move with the velocity of light. However, in our opinion this assumption is absolutely incorrect for an observer who would have reached the event horizon. We conclude that the event horizon can be reached only asymptotically in the Schwarzschild field and that this must also apply to the surface of a collapsing star.

The proper times are connected with the coordinate times via the metric factors

$$dT = \alpha a_R dt, \quad dT' = \alpha' a_R dt', \quad dT'' = \alpha'' a_R dt'', \quad a_R = \sqrt{1 - 2M/r}.$$

For an observer at rest is $v = 0$ and the Lorentz factor is $\alpha = 1$. For a freely falling observer who comes from infinity is known to be $v'' = -\sqrt{2M/r}$ and the Lorentz factor reads as

$$\alpha'' = 1\big/\sqrt{1 - v''^2} = 1\big/\sqrt{1 - 2M/r} = 1/a_R.$$

Thus one has

(4.6) $$dT = a_R dt, \quad dT' = \alpha' a_R dt', \quad dT'' = dt', \quad \alpha' = 1\big/\sqrt{1 - v'^2}.$$

The first term is the well-known expression for observers resting in the Schwarzschild field. The second term we have discussed in detail in [5] and we have set up a Lemaître transformation for this case as well. The third term shows that the proper time only coincides with the coordinate time if the observer comes from infinity. We owe Lemaître a detailed investigation for the last case.

With this we obtain from (4.5)

(4.7) $$ds^2 = dl^2 - a_R^2 dt^2 = dl'^2 - \alpha'^2 a_R^2 dt'^2 = dl''^2 - dt''^2.$$

Relating the metric of Weinberg to the surface of the star, we obtain in our abbreviated notation

$$ds^2 = dl'^2 - dT''^2.$$

This means that in the Weinberg line element two values are combined belonging to two different systems. dl' belongs to an observer who falls off from a finite position, dT'' to an observer who comes from infinity. With this it is evident that with (4.4) Einstein's law of velocity addition is violated and the Lorentz relations are destroyed, as well. Instead of (4.4) it should run as

(4.8) $$v_C = \frac{v_R - v_I}{1 - v_R v_I}, \quad \alpha_C = \alpha_R \alpha_I \left(1 - v_R v_I\right).$$

On the surface of the contracting star at the beginning of the collapse is $v_I^g = -r'_g/\mathcal{R}_0 = -\sqrt{2M/r'_g}$ a constant. If we write for this v_0, for the associated constant Lorentz factor α_0, and further $r'_g = r_g = r_0$, and if we simplify dl to the radial direction we get

$$\frac{dl}{dT'} = \alpha_c v_c, \quad dl = \alpha_R dr.$$

If we integrate the relation

$$dT' = \frac{\alpha_R}{\alpha_C v_C} dr = \frac{1}{\alpha_0 \left(v_R - v_0\right)} dr$$

and if we replace the outgoing coordinate r by an incoming $r_0 - r$ starting at r_0 we obtain for the proper time T' as a function of the distance from r_0 a progress which confirms with $T'(0) = 0$ and $T'(2M) = \infty$ that the surface of the star can only asymptotically approach the event horizon, if we comply with the laws of relativity. A plot of the function is shown in [5]. However, we are not aware of a collapsing model based on the relations (4.8).

With a further consideration we want to ensure the view that no collapsing star may fall below the event horizon, but that a dying star aims at an ultracompact state, which is very similar to a black hole, but is an object that can geometrically be described.

Since the exterior Schwarzschild solution has been proven and describes Nature well, one can assume that the interior solution can describe the interior of a star at least in a rough approximation. Although the two parameters, such as pressure and mass density are not sufficient to record the properties of a star, there is still hope that at least some basic properties of the model have general validity.

The interior Schwarzschild solution has a *pressure horizon*. If one reduces the volume of the stellar object, the pressure in the center will be infinitely high from a certain aperture angle of the spherical cap which geometrically describes the interior solution. The interior solution has a further horizon. If one drills a hole through the star along a diameter and letting fall an object through it, the object will oscillate through the star. Decreasing significantly the radius of the star the oscillating object would reach the velocity of light in the center of the star. This excludes a further reduction of the diameter of the star. We call this the *velocity horizon* [5]. It coincides with the pressure horizon. Unfortunately a collapsing version of the interior Schwarzschild solution is not known. But we can assume that at any given moment of the collapse the interior solution is a snapshot of the shrinking star. If we define the interior solution as a model for a contracting star then it basically follows from these properties that the star can only asymptotically contract to its inner horizon. However, this inner horizon lies higher on the Schwarzschild parabola than the event horizon. Thus, all the considerations about what could happen if the surface of a shrinking star or a falling observer is reaching the event horizon are superfluous. Therefore, if we assume the interior and exterior Schwarzschild solutions to describe Nature the formation of black holes is excluded.

Let us take a look at the conservation law. For a simple model with pressure and mass density the stress-energy-momentum tensor has the simple form

(4.9) $$T_{mn} = -p g_{mn} + \left(p + \mu_0\right) u_m u_n,$$

whereby the primes on the indices are omitted for the comoving system. In this system one has $u_m = \{0,0,0,1\}$ and thus $T_{\alpha 4} = 0,\ \alpha = 1,2,3$. From $T_{\alpha\ \|n}^{\ n} = 0$ one obtains

(4.10) $$p_{|\alpha} - \mu_0 E_\alpha = 0, \quad E_\alpha = \overset{4}{e}_4 \underset{4|\alpha}{e^4} .$$

Since one has $\overset{4}{e}_4 = 1$ for the above-discussed models, it follows $p_{|\alpha} = 0$. From the view of the comoving observer the pressure is spatially constant or zero. The latter is the case for the OS model and the Weinberg model. If one prefers a model with nonconstant or nonvanishing pressure one has to have $\overset{4}{e}_4 \neq 1$ and one has to take into account the structure of relations (4.6), especially the second expression.

Reinvestigating the two classical models we have demonstrated the difficulties that prevent the implementation of a physically relevant structure. If one wants to derive a model which takes into account the pressure inside, one has to fulfill some formal conditions. The time-like metric factor must be a function of the radial coordinate. Between the coordinate systems of comoving and non-comoving observers must mediate a Lemaître transformation, between the reference systems a Lorentz transformation, whereby Einstein's law of velocity addition must be considered. Moreover, descriptive geometric methods can be useful for the realization of these intentions.

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