Towards ellipsoidal representations of the Earth's gravitational field

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Introduction

• Traditionally, *spherical harmonics* (SH) are used for modelling the Earth’s gravitational field *globally*.

• These global basis functions are well suited when the generating body has a near *spherical surface*.

• SHs allow *easy computation* of the gravitational potential and its functionals.

• However, measured terrestrial and airborne gravity data are usually reduced to a rotating *reference ellipsoid*.

• Consequently, in such cases *ellipsoidal harmonics* (EH) are more appropriate than SHs.

• Since ellipsoidal basis functions include the geometry of the ellipsoid, the *approximation error* is expected to be less than for spherical basis functions.
Representation in Spherical Harmonics

\[ V(r, \lambda, \theta) = \frac{GM}{R} \sum_{n,m} \left( \frac{R}{r} \right)^{n+1} (C_{n,m}^s \cos(m\lambda) + S_{n,m}^s \sin(m\lambda)) \overline{P}_{n,m}(\cos \theta) \]

- \( GM \) = gravitational constant
- \( R \) = semi-major axis of the reference ellipsoid (or radius of the bounding sphere)
- \( r, \lambda, \theta \) = radial distance, spherical longitude, spherical co-latitude
- \( n, m \) = degree, order of the expansion
- \( \overline{P}_{n,m} \) = fully normalized associated Legendre function of the 1st kind
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- The ratio of the **geocentric parameters $R$ and $r$** defines the **upward continuation** property.
Representation in Spherical Harmonics

\[ V(r, \lambda, \theta) = \frac{GM}{R} \sum_{n,m} \left( \frac{R}{r} \right)^{n+1} \left( \tilde{C}_{n,m}^s \cos(m\lambda) + \tilde{S}_{n,m}^s \sin(m\lambda) \right) P_{n,m}(\cos \theta) \]

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- The ratio of the geocentric parameters \( R \) and \( r \) defines the upward continuation property.
- The harmonic coefficients \( \tilde{C}_{n,m}^s \) and \( \tilde{S}_{n,m}^s \) are the spherical harmonic coefficients.
Motivation for Ellipsoidal Modelling

- To get basis functions closer to a gravity measurement
- Less prior corrections or iterations needed
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- **Less prior corrections** or iterations needed

\[
\begin{align*}
u &= \text{semi-minor axis of confocal ellipsoids} \\
\lambda &= \text{spheroidal longitude} \\
\vartheta &= \text{reduced (spheroidal) co-latitude} \\
x &= \sqrt{u^2 + E^2} \sin \vartheta \cos \lambda = r \sin \theta \cos \lambda \\
y &= \sqrt{u^2 + E^2} \sin \vartheta \sin \lambda = r \sin \theta \sin \lambda \\
z &= u \cos \vartheta = r \cos \theta \\
E &= \sqrt{a^2 - b^2}
\end{align*}
\]
Motivation for Ellipsoidal Modelling

- To get basis functions **closer** to a gravity measurement
- **Less prior corrections** or iterations needed

In general, BVPs are ellipsoidal, if one or both requirements hold:
- **Orientation**: a reference direction is the ellipsoidal normal (to approximate plumbline's tangent)
- **Position**: gravity data are reduced to the ellipsoid
Motivation for Ellipsoidal Modelling

In general, BVPs are ellipsoidal, if one or both requirements hold:

- **Orientation**: a reference direction is the ellipsoidal normal (to approximate plumbline's tangent)
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Note, there are not many solutions that would solve the problem with both settings. For example, EGM2008 has used data on the ellipsoid, but the data were corrected to the spherical normal before the analysis (Pavlis 2012, JGR).
Representation in Ellipsoidal Harmonics

\[
V(u, \lambda, \vartheta) = \frac{GM}{a} \sum_{n,m} \frac{Q_{n,m}(\frac{u}{E})}{Q_{n,m}(\frac{b}{E})} (C_{n,m}^{e} \cos(m\lambda) + S_{n,m}^{e} \sin(m\lambda)) \overline{P}_{n,m}(\cos \vartheta)
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- The ratio of the associated Legendre functions of the 2\textsuperscript{nd} kind defines the **upward continuation** property.
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Definition of Local North Oriented Frames

- \( \text{LNOF}^s \)
- \( \text{LNOF}^e \)

taken from Sebera et al. (2012)
Representation in Ellipsoidal Harmonics

\[ V(u, \lambda, \vartheta) = \frac{GM}{a} \sum_{n,m} \frac{Q_{n,m}(\frac{u}{E})}{Q_{n,m}(\frac{b}{E})} \left( C_{n,m}^e \cos(m\lambda) + S_{n,m}^e \sin(m\lambda) \right) P_{n,m}(\cos \vartheta) \]

<table>
<thead>
<tr>
<th>Spherical</th>
<th>Ellipsoidal</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_x = -\frac{1}{r} V_\theta )</td>
<td>(- \frac{1}{L} V_\theta )</td>
</tr>
<tr>
<td>( V_y = -\frac{1}{r \sin \theta} V_\lambda )</td>
<td>(- \frac{1}{v \sin \vartheta} V_\lambda )</td>
</tr>
<tr>
<td>( V_z = V_r )</td>
<td>( \frac{v}{L} V_u )</td>
</tr>
<tr>
<td>( V_{xx} = \frac{1}{r} V_r + \frac{1}{r^2} V_\theta \theta )</td>
<td>( \frac{u v^2}{L^4} V_u + \frac{1}{L^2} V_\theta \theta + \frac{E^2 \cos \vartheta \sin \vartheta}{L^4} V_\theta )</td>
</tr>
<tr>
<td>( V_{yy} = \frac{1}{r} V_r + \frac{1}{r^2 \sin^2 \theta} V_\lambda \lambda + \frac{\coth \theta}{r^2} V_\theta )</td>
<td>( \frac{u}{L^2} V_u + \frac{1}{v^2 \sin^2 \vartheta} V_\lambda \lambda + \frac{\cot \vartheta}{L^2} V_\vartheta )</td>
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<tr>
<td>( V_{zz} = V_{rr} )</td>
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<td>( V_{xy} = \frac{1}{r^2 \sin \theta} V_\theta \lambda - \frac{\cos \theta}{r^2 \sin^2 \theta} V_\lambda )</td>
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Cartesian derivatives in the LNOFs; taken from R. Koop (1993)
Multi-Resolution Representation (MRR)

- The representation of the gravity field in terms of spherical or ellipsoidal harmonics is only appropriate if the input data is
  - distributed homogeneously over the globe and of
  - similar accuracy.
- Gravity field modelling in terms of spherical (radial) basis functions means an appropriate alternative and is nowadays successfully used in regional applications.
- The application of the wavelet transform allows the decomposition of a signal $F$ into a certain number of detail signals $G_j$ each related to a specific frequency band (resolution level $j \in \mathbb{N}_0$).
- This decomposition means a multi-resolution representation (MRR).
- Here we discuss the MRR briefly in the context of ellipsoidal modelling; for details see, e.g., Schmidt and Fabert (2008)
The different data sets

- are sensitive to specific frequency bands and
- have a spatial resolution depending on the measurement campaign.

Thus, they are related to well defined resolution levels within the MRR.
Pyramid Algorithm

observation technique 1 → $d_J$ → $G_J$ ...
observation technique 2 → $d_{J-1}$ → $G_{J-1}$ ...
observation technique 3 → $d_{J-2}$ → $G_{J-2}$ ...
observation technique .. → $d_{j'}$ → $G_{j'}$ ...

$d_{j-1} = H_{j-1} d_j$
$d_{j-2} = H_{j-2} d_{j-1}$
$d_{j-3} = H_{j-3} d_{j-2}$
Pyramid Algorithm

Different detail signals $G_j, G_{j-1}, \ldots, G_{j'}$ can be computed from different observation techniques considering the low-pass filter matrices $H_{j-1}, \ldots$.
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Multi-Resolution Representation (MRR)

- Formulation of the MRR for \( F(u, \lambda, \vartheta) := V(u, \lambda, \vartheta) - V_{\text{back}}(u, \lambda, \vartheta) \) or functionals such as gravity anomalies, gravity gradients, etc.:

\[
F(u, \lambda, \vartheta) \approx F_{J+1}(u, \lambda, \vartheta) = F_{j'}(u, \lambda, \vartheta) + \sum_{j=j'}^J G_j(u, \lambda, \vartheta)
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- \( F_{j'}(u, \lambda, \vartheta) = \) low-pass filtered signal

- \( G_j(u, \lambda, \vartheta) = \) band-pass filtered detail signal
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\psi_j(P, P') = \sum_{n=0}^{N_j} \sum_{m=-n}^{n} \frac{Q_{n,|m|}(\frac{u}{E})}{Q_{n,|m|}(\frac{b}{E})} \Psi_{j;n,m} Y_{n,m}(P') Y_{n,m}(P)
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- The Legendre coefficients \( \Psi_{j;n,m} \) define the shape of the level-\( j \) wavelet function \( \psi_j(P, P') \) depending on the two points \( P(u, \lambda, \vartheta) \) and \( P'(b, \lambda', \vartheta') \).

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- For $\Psi_{j;n,m} = 1$ and $N_j \to \infty$ we get the Poisson kernel.

$$\psi_j(P, P') = \sum_{n=0}^{N_j} \sum_{m=-n}^{n} \frac{Q_{n,|m|}(\frac{u}{E})}{Q_{n,|m|}(\frac{b}{E})} \Psi_{j;n,m} Y_{n,m}(P') Y_{n,m}(P)$$
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- The level-\( j \) wavelet function \( \psi_j(P, P') \) depends on the ratio of the associated Legendre functions of 2\(^{nd} \) kind (ALF 2).

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\psi_j(P, P') = \sum_{n=0}^{N_j} \sum_{m=-n}^{n} \frac{Q_n, |m| \left( \frac{u}{E} \right)}{Q_n, |m| \left( \frac{b}{E} \right)} \Psi_{j;n,m} Y_{n,m}(P') Y_{n,m}(P)
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- The level-\( j \) wavelet function \( \psi_j(P, P') \) depends on the ratio of the associated Legendre functions of 2\(^{\text{nd}}\) kind (ALF 2).

- For terrestrial measurements with a resolution of 2\(\text{'}\) we setup the highest resolution level to \( J = 12 \), thus the maximum degree value is \( N_J = 4095 \).

\[
\psi_j(P, P') = \sum_{n=0}^{N_j} \sum_{m=-n}^{n} \frac{Q_{n,|m|}(\frac{u}{E})}{Q_{n,|m|}(\frac{b}{E})} \psi_{j;n,m} Y_{n,m}(P') Y_{n,m}(P)
\]
Associated Legendre Functions of 2\textsuperscript{nd} kind

- For the computation of the ALF 2 we use the hypergeometric formulation with Jekeli's renormalization.
- Thus, the ALFs 2 are replaced by Jekeli’s function

\[ S^0_{n,m} \left( \frac{u}{E} \right) = \left( 1 + \frac{E^2}{u^2} \right)^{\frac{m}{2}} \left( \frac{R}{u} \right)^{n+1} {}_{2}F_{1} \left( \frac{n + m + 2}{2}, \frac{n + m + 1}{2}, n + \frac{3}{2}, -\frac{E^2}{u^2} \right) \]

- can be computed in a straightforward recurrence (Jekeli, 1988)
Associated Legendre Functions of 2\textsuperscript{nd} kind

\[ V(u, \lambda, \vartheta) = \frac{GM}{a} \sum_{n,m} \frac{\bar{S}^0_{n,m}(u/E)}{\bar{S}^0_{n,m}(b/E)} (\bar{C}^e_{n,m} \cos(m\lambda) + \bar{S}^e_{n,m} \sin(m\lambda)) \bar{P}_{n,m}(\cos \vartheta) \]

\[ \bar{S}^0_{n,m}(u/E) = \left(1 + \frac{E^2}{u^2}\right)^{\frac{m}{2}} \left(\frac{R}{u}\right)^{n+1} _2F_1 \left(\frac{n+m+2}{2}, \frac{n+m+1}{2}, n + \frac{3}{2}, -\frac{E^2}{u^2}\right) \]

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- the same function as above but with faster convergence (Sebera et al., 2012)
Associated Legendre Functions of 2\textsuperscript{nd} kind

$$V(u, \lambda, \vartheta) = \frac{GM}{a} \sum_{n,m} \frac{\bar{S}_{n,m}^0(u/E)}{\bar{S}_{n,m}^0(b/E)} \left( \overline{C}_{n,m}^e \cos(m\lambda) + \overline{S}_{n,m}^e \sin(m\lambda) \right) \overline{P}_{n,m}(\cos \vartheta)$$

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- $\bar{S}_{n,m}^0(u/E) = \left(\frac{R}{\sqrt{u^2 + E^2}}\right)^{n+1} \, _2F_1\left(\frac{n+m+1}{2}, \frac{n-m+1}{2}, n+\frac{3}{2}, -\frac{E^2}{u^2 + E^2}\right)$

  - the same but much better suitable for high-degree computations; $n > 3000$. 
**Associated Legendre Functions of 2\textsuperscript{nd} kind**

\[
V(u, \lambda, \vartheta) = \frac{GM}{a} \sum_{n,m} \frac{\tilde{S}_{n,m}^0(u/E)}{\tilde{S}_{n,m}^0(b/E)} \left( C_{n,m}^e \cos(m\lambda) + S_{n,m}^e \sin(m\lambda) \right) P_{n,m}(\cos \vartheta)
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Associated Legendre Functions of 2\textsuperscript{nd} kind

\[ \psi_j(P, P') = \sum_{n=0}^{N_j} \sum_{m=-n}^{n} \frac{S^0_{n|m}}{S^0_{n|m}} \left( \frac{u}{E} \right) \Psi_{j;n,m} Y_{n,m}(P') Y_{n,m}(P) \]

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Summary

- From a wide range of approaches for modelling the Earth’s gravitational field we presented two *ellipsoidal* ones:
  - a *global* representation based on ellipsoidal harmonics,
  - a *regional* multi-resolution representation based on ellipsoidal basis functions.
- The chosen approach depends on the *distribution*, the *variety* and the *sensitivity* of the input data sets.
- The presented basis functions require the evaluation of the *associated Legendre functions* of the 2nd kind.
- We use the *hypergeometric formulation* and Jekeli’s renormalization for computing the associated Legendre functions of 2nd kind.
- One advantage of this procedure is that the functions \( \text{^*}S_{n,m}^0 \) are also needed for the *transformation* between SH and EH coefficients.
References

Backup slides
Associated Legendre Functions of 2\textsuperscript{nd} kind

- For the computation of the ALFs of the 2\textsuperscript{nd} kind we use the \textbf{hypergeometric formulation} with \textit{Jekeli's renormalization}

\[
\bar{S}^{0}_{n,m} \left( \frac{u}{E} \right) = \left( \frac{R}{E} \right)^{n+1} \frac{i^{n+1}(2n+1)!}{2^n n!} \sqrt{\frac{\epsilon_m}{(2n+1)(n-m)!(n+m)!}} \bar{Q}_{n,m}(z)
\]

\[
\bar{Q}_{n,m}(z) = (-1)^m \frac{2^n n!(n+m)!}{(2n+1)!} \left( \frac{z^2-1}{z^{n+m+1}} \right)^{\frac{m}{2}} 2F_1 \left( \frac{n+m+2}{2}, \frac{n+m+1}{2}, n + \frac{3}{2}, \frac{1}{z^2} \right) \sqrt{\frac{(2n+1)(n-m)!}{\epsilon_m(n+m)!}}
\]

\textit{Gauss hypergeometric function} \hspace{1cm} \textit{Normalization}

- This gives \textit{Jekeli’s function} as

\[
\bar{S}^{0}_{n,m} \left( \frac{u}{E} \right) = \left( 1 + \frac{E^2}{u^2} \right)^{\frac{m}{2}} \left( \frac{R}{u} \right)^{n+1} 2F_1 \left( \frac{n+m+2}{2}, \frac{n+m+1}{2}, n + \frac{3}{2}, -\frac{E^2}{u^2} \right)
\]
Representation in Ellipsoidal Harmonics

\[ V(u, \lambda, \vartheta) = \frac{GM}{a} \sum_{n,m} \frac{S^0_{n,m}(u/E)}{S^0_{n,m}(b/E)} \left( C^e_{n,m} \cos(m\lambda) + S^e_{n,m} \sin(m\lambda) \right) P_{n,m}(\cos \vartheta) \]